Categories, Symmetry and Manifolds

Math 4800, Fall 2020

3. Metric Spaces. A **metric space** is a set M together with a **metric** d that measures the distance between two points of M. That is, d is a function:

$$d : M \times M \to [0, \infty) = \mathbb{R}^{\geq 0}$$

that is required to satisfy the following:

(0) $d(p, p) = 0$ for all $p \in M$.

(1) $d(p, q) > 0$ whenever $p \neq q$ (social distancing).

(2) $d(p, q) = d(q, p)$ for all p and q (distance is observer-independent).

(3) The *triangle inequality* $d(p, r) \leq d(p, q) + d(q, r)$ for all $p, q, r \in M$.

Examples. (a) $\mathbb{R}^n$ is a metric space with many different notions of distance.

If $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$, then each of the following is a metric:

(Max distance)

$$d_{\max}(p, q) = \max_{i=1,...,n} \{|p_i - q_i|\}$$

(Manhattan distance)

$$d_{\mathrm{Man}}(p, q) = \sum_{i=1}^{n} |p_i - q_i|$$

(Euclidean distance)

$$d(p, q) = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

(b) The two-sphere has a great circle (spherical) metric:

$$S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

Each pair of distinct points $p, q \in S^2$ determines a great circle through p and q and $d(p, q)$ is the arc length of the (smaller) of the arcs of the great circle with endpoints p and q. (This generalizes to a great circle metric on the n-sphere).

(c) Metric graphs. If Γ is a connected graph with edge set E, then an assignment

$$l : E \to \mathbb{R}^{>0}$$

of a length to each edge makes Γ into a **metric graph**, in which:

$$d(p, q) = \text{length of the shortest path from } p \text{ to } q$$

This is a metric on the set V of **vertices** of the graph. With some thought, one can replace the edges of the graph with line segments of the given lengths and obtain a shortest path distance function for pairs of points on the "tinkertoy" structure consisting of the edge segments joined at the vertices.

Remark. This last example is a discrete analogue of a **Riemannian metric** on a manifold, with respect to which one defines lengths of paths between two points. The shortest paths between points are *geodesics* and their lengths define a distance function on the manifold. Come to think of it, (b) is an example of this, as is the Euclidean distance, in which the shortest paths are arcs of great circles and line segments, respectively.

Given two metric spaces (M, d) and (N, e),

Definition 3.1. A function $f : (M, d) \to (N, e)$ is **distance decreasing** if:

$$d(p, q) \geq e(f(p), f(q)) \text{ for all } p, q \in M$$

Examples. (a) The projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ given by $\pi(x_1, ..., x_{n+1}) = (x_1, ..., x_n)$ is distance decreasing (and linear). It is not a bijection, so it has no two-sided inverse, but the "zero section" of the projection $z(x_1, ..., x_n) = (x_1, ..., x_n, 0)$ is distance preserving and a *right* inverse of π, in the sense that $\pi \circ z = 1_{\mathbb{R}^n}$.

(b) A strictly distance decreasing function can be a bijection. The function:

$$f(x_1, ..., x_n) = \left(\frac{1}{2}x_1, ..., \frac{1}{2}x_n\right)$$

from $\mathbb{R}^n$ to $\mathbb{R}^n$ scales distances by $1/2$ and has a (distance increasing) inverse.

(c) The identity $1_{\mathbb{R}^n}$ is a distance decreasing function from $(\mathbb{R}^n, d)$ to $(\mathbb{R}^n, d_{\max})$, since the max distance is always less than (or equal to) the Euclidean distance.

Metric Spaces. The collection of metric spaces with distance decreasing functions is a category, which we will denote by $\mathfrak{Met}$. One needs to check that:

(i) The identity function $1_{(X,d)}$ is distance decreasing from (X, d) to (X, d).

(ii) The composition of distance decreasing functions is distance decreasing.

but these are easily seen to be true.

Definition 3.2. The isomorphisms in $\mathfrak{Met}$ are the distance-preserving bijections, also called **isometries**, and the symmetries of X are the isometries from X to itself.

Any property of a metric space that is defined solely in terms of the metric is shared by any isometric space. Since Cauchy sequences and convergent sequences are defined in terms of the metric, the following is an example:

Definition 3.3. A metric space X is **complete** if every Cauchy sequence of points in X converges to a (unique) point $x \in X$.

Example. Remove the origin from $\mathbb{R}^n$ but retain the Euclidean metric. This is not a complete metric space because we punched a hole in it. On the other hand $\mathbb{R}^n$ is complete, so there is no isometry from $\mathbb{R}^n - \{\text{origin}\}$ to $\mathbb{R}^n$. The open disk:

$$D = \{(x, y) \mid x^2 + y^2 < 1\} \subset \mathbb{R}^2$$

is also not complete for the Euclidean metric. However there is an interesting *hyperbolic (Poincaré) metric* on D, with respect to which it is complete. In this metric the geodesics (shortest paths) are arcs of circles that meet the unit circle bounding D at right angles. But unlike the arcs of great circles on the sphere, this distance is not the ordinary length of the arc.

Symmetries of $\mathbb{R}^n$**.** A symmetry of $(\mathbb{R}^n, d)$ is also called a rigid motion.
Our aim is to "classify" all of these. Let's start with the symmetries of $(\mathbb{R}, |\cdot|)$.

(i) Translation by $r \in \mathbb{R}$. This is the symmetry $\tau_r(x) = x + r$.

(ii) Reflection across $s \in \mathbb{R}$. This is the symmetry $\rho_s(x) = -x + 2s$.

It is an easy exercise (and intuitive) to see that translations and reflections are symmetries of $(\mathbb{R}, |\cdot|)$. Notice that a translation has no fixed points (except for τ_0). Each reflection ρ_s on the other hand fixes the single point s. Moreover,

$$\tau_{r_1} \circ \tau_{r_2} = \tau_{r_1+r_2} \text{ and } \rho_s \circ \rho_s = \tau_0$$

More generally, the composition of two reflections is a translation:

$$\rho_{s_1} \circ \rho_{s_2}(x) = \rho_{s_1}(-x + 2s_2) = (x - 2s_2) + 2s_1 = x + 2(s_1 - s_2)$$

and the composition of two different reflections does not commute!

Proposition 3.4. Every symmetry of $\mathbb{R}$ is a translation or a reflection.

Proof. We prove this by "deconstructing" the symmetry $f : \mathbb{R} \to \mathbb{R}$.

(1) Let $f(0) = r$. Then by composing with the translation τ_{-r} we obtain:

$$\phi = \tau_{-r} \circ f \text{ with } \phi(0) = \tau_{-r}(f(0)) = \tau_{-r}(r) = 0$$

which is a symmetry of $\mathbb{R}$ fixing the origin.

(2) A symmetry of $\mathbb{R}$ that fixes the origin is either $\phi(x) = x$ or $\phi(x) = -x$.

A symmetry ϕ that fixes the origin must satisfy:

$$1 = |1 - 0| = |\phi(1) - \phi(0)| = |\phi(1)|$$

so either $\phi(1) = 1$ or $\phi(1) = -1$. More generally,

$$|x| = |x - 0| = |\phi(x) - 0| = |\phi(x)|$$

so either $\phi(x) = x$ or $\phi(x) = -x$. But if $\phi(1) = 1$, it is easy to see that $\phi(x) = x$ for **all** x, and if $\phi(1) = -1$, then it is easy to see again that $\phi(x) = -x$ for **all** x. Thus either:

$$\phi(x) = x \text{ and } f(x) = \tau_r(x) = r + x$$

or else

$$\phi(x) = -x \text{ and } f(x) = \tau_r(-x) = r - x$$

i.e. $f(x)$ is either the translation by r or else the reflection across $r/2$. □

To handle symmetries of $\mathbb{R}^n$ in the same way, we introduce:

Definition 3.5. Given vectors $\vec{v} = (v_1, ..., v_n)$ and $\vec{w} = (w_1, ..., w_n)$ in $\mathbb{R}^n$,

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots v_n w_n \in \mathbb{R}$$

is their **dot product**. This is:

(a) Commutative and Bilinear ($\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ and $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$). And

(b) Computes the Euclidean length of the vector $\vec{v}$ via $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$

(c) Computes the angle θ between vectors $\vec{v}$ and $\vec{w}$ via:

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$$

Definition 3.6. $\vec{e}_i$ are the unit vectors $(0, ..., 1, .., 0)$ (with 1 in the ith position).

Proposition 3.7. Write $|p - q| = d(p, q)$ for the length of the vector from p to q.

(a) If $\vec{v}_1, ..., \vec{v}_n \in \mathbb{R}^n$ are mutually perpendicular unit vectors, then:

$$\phi(x_1, ..., x_n) = x_1 \vec{v}_1 + \cdots + \cdots x_n \vec{v}_n$$

is a symmetry of $\mathbb{R}^n$ that fixes the origin.

(b) The identity is the only symmetry that fixes the origin and each $\vec{e}_1, ..., \vec{e}_n$.

Proof. (a) The function ϕ is *linear* by construction, so $\phi(0) = 0$ and:

$$|\phi(x_1, ..., x_n) - \phi(y_1, ..., y_n)| = |\phi(x_1 - y_1, ..., x_n - y_n)|$$

and we may conclude that ϕ is a symmetry once we show that:

$$|\phi(x_1, ..., x_n)| = |(x_1, ..., x_n)| \text{ for all } (x_1, ..., x_n) \in \mathbb{R}^n$$

But because $\vec{v}_1, ..., \vec{v}_n$ are unit vectors and mutually perpendicular, we have:

$$|\phi(x_1, ..., x_n)|^2 = (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) \cdot (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n)$$
$$= x_1^2 + \cdots + x_n^2 = |(x_1, ..., x_n)|^2$$

which gives (a). When we arrange the vectors $\vec{v}_i$ as columns in a matrix, then:

$$\phi(x_1, ..., x_n) = [\vec{v}_1 \cdots \vec{v}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

But the inverse of this matrix is given by the vectors $\vec{v}_i$ arranged as **rows**:

$$\phi^{-1}(y_1, ..., y_n) = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

i.e. the transpose of this matrix is the inverse. It follows that ϕ is a bijection, and also, amusingly, that the rows of the original matrix are another set of mutually perpendicular unit vectors since the inverse matrix is also a symmetry.

(b) We prove this by brute force. If f is a symmetry that fixes 0 and $\vec{e}_1$, let $(y_1, ..., y_n) = f(x_1, ..., x_n)$. Then:

$$y_1^2 + \cdots + y_n^2 = x_1^2 + \cdots + x_n^2$$
$$(y_1 - 1)^2 + y_2^2 + \cdots + y_n^2 = (x_1 - 1)^2 + x_2^2 + \cdots + x_n^2$$

and we conclude that $y_1 = x_1$. In the same way, if f fixes 0 and $\vec{e}_i$, then $y_i = x_i$. □

Corollary 3.8. If ϕ is a symmetry of $\mathbb{R}^n$ that fixes 0, let $\vec{v}_i = \phi(\vec{e}_i)$. Then:

$$\phi(x_1, ..., x_n) = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

is the symmetry in (a) above. Thus every symmetry that fixes the origin is linear.

Example. Let ϕ be a symmetry of $\mathbb{R}^2$ that fixes the origin. Then:

$$\phi(1, 0) = (\cos(\theta), \sin(\theta)) \text{ for some angle } \theta, \text{and then}$$
$$\phi(0, 1) = (-\sin(\theta), \cos(\theta)) \text{ or } \phi(0, 1) = (\sin(\theta), -\cos(\theta))$$

because these are the only two unit vectors that are perpendicular to $(\cos(\theta), \sin(\theta))$. In the first case,

$$\phi_\theta(x, y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is the **rotation** by θ (counterclockwise) and in the second case,

$$\rho_{\theta/2}(x, y) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is the **reflection** across the line $y = \tan(\theta/2)x$.

Definition 3.9. A linear symmetry ϕ of $\mathbb{R}^n$ is an **orthogonal transformation**.

Remark. The **determinant** of the matrix distinguishes rotations from reflections. A rotation has determinant $+1$ and a reflection has determinant -1. Rotations preserve the "orientation" of the plane and reflections switch the orientation. In §4 we will see that the determinant of an orthogonal transformation is always ± 1 and we will use the characteristic polynomial of an orthogonal transformation to prove:

Theorem 3.10. Every orientation-preserving orthogonal transformation ϕ of $\mathbb{R}^3$ is a rotation about a (unit) vector $\vec{v} \in \mathbb{R}^3$ that is **fixed** by ϕ.

As for symmetries of $\mathbb{R}^n$, we can say the following:

Proposition 3.11. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetry and $f(0) = r \in \mathbb{R}^n$, then:

$$f(p) = r + \phi(p)$$

for some orthogonal transformation ϕ.

This is the exact analogue of Proposition 3.4. The only difference is that there are a wealth of symmetries other than translations and reflections when $n > 1$.

Remark. The composition of $f(p) = r + \phi(p)$ and $g(p) = s + \psi(p)$ is:

$$(f \circ g)(p) = r + \phi(s + \psi(p)) = (r + \psi(s)) + (\phi \circ \psi)(p)$$

since ϕ is linear and the **inverse** of $f(p) = r + \phi(p)$ is $f^{-1}(q) = \phi(-r) + \phi^{-1}(q)$

Symmetries of Regular Polyhedra.

Definition 3.12. The regular inscribed n-gon P_n is the polygon in $\mathbb{R}^2$ with vertices:

$$p_m = e^{2\pi i m/n} = \left(\cos\left(\frac{2\pi m}{n}\right), \sin\left(\frac{2\pi m}{n}\right)\right); \ m = 1, ..., n$$

i.e. the vertices are regularly spaced around the unit circle.

Proposition 3.13. There are $2n$ symmetries of P_n.

A symmetry of P_n fixes the origin, since the origin is the only point of $\mathbb{R}^2$ that is at a unit distance from all the vertices. Thus it is either a **rotation** or a **reflection**. The rotations and reflections that fix the polygon are:

$$\phi_\theta \text{ and } \ \rho_{\theta/2} \text{ for } \theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, ..., \frac{(2n-2)\pi}{n}$$

The **rotations** alone form a cyclic group which, in this context, we denote by:

$$C_n = \text{ the rotational symmetries of } P_n$$

and the collection of all symmetries, we denote by:

$$D_{2n} = \text{ rotational symmetries and reflections of } P_n$$

We can completely describe the symmetries by specifying their compositions:

(1) $\phi_{\theta_1} \circ \phi_{\theta_2} = \phi_{\theta_1+\theta_2}$ (2) $\rho_{\theta_1/2} \circ \rho_{\theta_2/2} = \phi_{\theta_2-\theta_1}$

(3) $\phi_{\theta_1} \circ \rho_{\theta_2/2} = \rho_{(\theta_1+\theta_2)/2}$ (4) $\rho_{\theta_1/2} \circ \phi_{\theta_2} = \rho_{(\theta_1-\theta_2)/2}$

This is often written abstractly, letting:

$x = \phi_{\frac{2\pi}{n}}$ be the generator of C_n and $y = \rho_0$ be the reflection across the x-axis

from which we conclude that $x^m = \phi_{2\pi m/n}$ and $x^m y = \rho_{2\pi m/n}$ and, finally:

$$D_{2n} = \{1, x, x^2, ..., x^{n-1}, y, xy, x^2y, , , , mx^{n-1}y \mid x^n = 1, \ y^2 = 1 \text{ and } yx = x^{-1}y\}$$

which is enough information to completely specify D_{2n} and its multiplication.

Definition 3.14. The **platonic solids** are the five regular solids:

The Tetrahedron (4 triangular faces, 6 edges, 4 vertices)

The Heptahedron, aka The Cube (6 square faces, 12 edges, 8 vertices)

The Octahedron (8 triangular faces, 12 edges, 6 vertices)

The Dodecahedron (12 pentagonal faces, 20 edges, 20 vertices)

The Icosahedron (20 triangular faces, 30 edges, 12 vertices)

which may be inscribed in the unit sphere.

These five solids come in dual pairs, from the point of view of their symmetries, since a symmetry of a solid S is a symmetry of the dual solid T whose vertices are the *midpoints* of the faces of S. The vertices of T correspond to the faces of S and the faces of T correspond to the vertices of S. The tetrahedron is dual to itself, but

$$S = \text{ Heptahedron } \leftrightarrow\ T = \text{Octahedron},\ \ S = \text{ Dodecahedron } \leftrightarrow\ T = \text{Icosahedron}$$

Using Theorem 3.10, we can tabulate the orientation-preserving symmetries of the regular solids S. These are analogous to the rotations of a regular polygon.

The Tetrahedron. The symmetries rotate about:

(1) 4 lines through a vertex and the midpoint of an opposite face (three rotations)

(2) 3 lines through the midpoints of opposite edges (two rotations)

One of each set of rotations is the identity, so this gives:

$$1 + 2 \times 4 + 1 \times 3 = 12 \text{ symmetries}$$

These symmetries do not commute. They form a non-Abelian group (see §5) that is isomorphic to the *alternating group* A_4 of even permutations of [4]. One can see this by considering how these symmetries permute the 4 vertices of the tetrahedron.

The Heptahedron. The symmetries rotate about:

(1) 4 diagonals joining opposite vertices (three rotations)

(2) 3 lines through the midpoints of opposite faces (four rotations)

(3) 6 lines through the midpoints of opposite edges (two rotations)

Altogether this gives

$$1 + 2 \times 4 + 3 \times 3 + 1 \times 6 = 24 \text{ symmetries}$$

forming a non-Abelian group isomorphic to the group S_4 of **all** permutations of [4]. One sees this by permuting the 4 diagonals of the cube with these symmetries.

The Dodecahedron. The symmetries rotate about:

(1) 10 lines joining opposite vertices (three rotations)

(2) 6 lines joining the midpoints of opposite faces (five rotations)

(3) 15 lines joining the midpoints of opposite edges (two rotations)

Giving us a total of:

$$1 + 2 \times 10 + 4 \times 6 + 1 \times 15 = 60 \text{ symmetries}$$

which is isomorphic to the alternating group A_5 of even permutations of [5]. It's an interesting question to find the five things that are permuted by the symmetries.

Assignment 3.

1. (a) The product of two metric space (X_1, d_1) and (X_2, d_2) in the category $\mathfrak{Met}$ is the Cartesian product $X_1 \times X_2$. What is the metric?

(b) Show that there is no coproduct of metric spaces, because in $X_1 \sqcup X_2$ the points of X_1 would need to be infinitely far from points of X_2 in order to satisfy the universal property.

2. The standard unit ball in $\mathbb{R}^n$ is the set of points $p \in \mathbb{R}^n$ such that:

$$d(0, p) < 1 \text{ where } 0 \in \mathbb{R}^n \text{ is the origin}$$

Describe the unit balls in $\mathbb{R}^2$ and in $\mathbb{R}^3$ for each of the following metrics:

(a) The max distance metrix

(b) The Manhattan metric

(c) The Euclidean metric

3. Find a graph with six vertices and unit edge lengths in which:

- nine pairs of vertices v, w satisfy $d(v, w) = 1$, and
- six pairs of vertices v, w satisfy $d(v, w) = 2$.

4. Check that $f^{-1}(q) = \phi^{-1}(-r) + \phi^{-1}(q)$ is a two-sided inverse of $f(p) = r + \phi(p)$.

5. (a) The reflection across the line $y = mx + b$ is a symmetry of $\mathbb{R}^2$. Express it in the form $f(p) = r + \phi(p)$.

(b) Rotation by an angle θ around a point $c \in \mathbb{R}$ is a symmetry of $\mathbb{R}^2$. Express it in the form $f(p) = r + \phi(p)$.

6. Verify the compositions (1)-(4) for the dihedral group D_{2n}.

7. Verify that D_6 is the same as the permutations of [3]. Verify the equality:

$$yx^k y^{-1} = x^{-k}$$

for all the dihedral groups D_{2n}. (We'll be using this later....)

Possible Projects.

3.1. Explore the hyperbolic metric on the unit disk. Find the distance between points of the unit disk for this metric. Check that the triangle inequality holds and explain why the metric is complete.

3.2. Verify that there are only five regular solids and carefully check that their symmetries are A_4, S_4 and A_5, as advertised.

3.3. What are the symmetries of the tesseract (four-dimensional cube)?