
A projective variety over \( k \) is obtained from a \( \mathbb{Z} \)-graded \( k \)-algebra domain \( A \) (via the functor \( \text{maxproj} \)) analogously to the realization of an affine variety from an \( k \)-algebra (ungraded) domain \( A \) (via the functor \( \text{maxspec} \)). The key difference is that unlike the affine case, in which the domain is recovered from the regular functions, the only regular functions on a projective variety are the constants.

**Definition 4.1.** As a set, projective space \( \mathbb{P}^n_k \) is the locus of lines through \( 0 \in k^{n+1} \).

**Definition 4.2.** The polynomial ring graded by degree:

\[
S_\bullet = \bigoplus_{d=0}^{\infty} k[x_0, \ldots, x_n]_d
\]

is defined by

\[
S_d = \left\{ \sum_{|I|=d} c_I x_I \mid x_I = x_0^{i_0} \cdots x_n^{i_n}, c_I \in k \right\}
\]

i.e. \( S_d \) is the vector space of homogeneous polynomials of degree \( d \), with:

\[
S_d \cdot S_e \subset S_{d+e}
\]

**Definition 4.3.** An ideal \( I \subset S_\bullet \) is homogeneous if:

\[
I = \bigoplus_{d=0}^{\infty} I \cap k[x_0, \ldots, x_n]_d,
\]

and in that case we let \( I_d = I \cap k[x_0, \ldots, x_n]_d \)

i.e. \( I \) is generated by (finitely many!) homogeneous polynomials, so that

\[
f = f_0 + \cdots + f_d \in I \leftrightarrow f_e \in I_e \text{ for all } e
\]

The quotient by a homogeneous ideal is a graded ring:

\[
S_\bullet / I = A_\bullet \text{ with } A_d = S_d / I_d \text{ and } A_d \cdot A_e \subset A_{d+e}
\]

**Example.** (a) The irrelevant homogeneous maximal ideal in \( S_\bullet \) is:

\[
S_+ = \bigoplus_{d=1}^{\infty} k[x_0, \ldots, x_n]_d = \langle x_0, \ldots, x_n \rangle
\]

This ideal contains all homogeneous ideals in \( S_\bullet \) other than the ideal \( \langle 1 \rangle \).

(b) If \( X \subset \mathbb{P}^n_k \), then the affine cone over \( X \) is:

\[
C(X) = \{ (a_0, \ldots, a_n) \in k^{n+1} \mid k \cdot (a_0, \ldots, a_n) \in X \} \cup \{ (0, \ldots, 0) \}
\]

The ideal \( I(X) := I(C(X)) \subset S_\bullet \) is a homogeneous ideal (if \( k \) is infinite), and:

\[
k[X]_\bullet = k[x_0, \ldots, x_n]_\bullet / I
\]

(with this convention, \( I(\emptyset) = S_+ \), though one could argue for \( I(\emptyset) = \langle 1 \rangle \))

(c) For a homogeneous ideal \( I \subset S_+ \),

\[
X(I) = C(X) \subset k^{n+1}
\]

is an affine cone over some \( X \subset \mathbb{P}^n_k \) and we let \( X := X(I) \subset \mathbb{P}^n_k \) be the associated algebraic subset of \( \mathbb{P}^n_k \).

This sets up a version of the Nullstellensatz for radical homogeneous ideals:
The Projective Nullstellensatz. The radical homogeneous ideals $I \subset S_+$ are in bijection with the algebraic subsets $X = X(I) \subset \mathbb{P}^n_k$ via the mappings $X$ and $I$, with the prime ideals corresponding to irreducible algebraic sets and the maximal prime ideals properly contained in $S_+$ corresponding to the points $x \in \mathbb{P}^n_k$ via:

$$m_x = \langle a_jx_i - a_ix_j \rangle \text{ for } x = k \cdot (a_0, \ldots, a_n)$$

Proof. This follows from the ordinary Nullstellensatz applied to affine cones and the fact that $\text{rad}(I)$ is a homogeneous ideal when $I$ is a homogeneous ideal.

Projective Coordinates. We will write $x \in \mathbb{P}^n_k$ in coordinates as the ratio:

$$(a_0 : \cdots : a_n)$$

with the understanding that $(a_0 : \cdots : a_n) = (\lambda a_0 : \cdots : \lambda a_n)$ for $\lambda \in k^*$.

Remark. If $F \in S_d$ is homogeneous of degree $d$, then:

$$F(\lambda a_0 : \ldots : \lambda a_n) = \lambda^d F(a_0 : \cdots : a_n)$$

so although the value $F(x)$ is not well-defined, it does make sense to say $F(x) = 0$. When $F$ is not homogeneous, even this statement is not well-defined.

Example. In the projective space $\mathbb{P}^n_k$ of $n \times n$ matrices,

$X(\Delta)$ is the locus (hypersurface) of singular matrices where $\Delta \in S_n$ is the determinant polynomial. The complement is $\text{PGL}(n, k)$.

The following Lemma is useful.

Lemma 4.4. For a homogeneous ideal $I \subset S_+$,

$$X(I) = \emptyset \Leftrightarrow S_+ \subseteq \text{rad}(I) \Leftrightarrow S_d \subset X(I) \text{ for some } d$$

Proof. The first equivalence is immediate, and if $S_+ \subseteq \text{rad}(I)$, then

$$x_i^{d_i} \in I \text{ for some } d_0, \ldots, d_n$$

and then $S_d \subset I$ for $d > (d_0 + \cdots + d_n) - n$. The converse is clear. \(\square\)

We now enlarge our stable of $\mathbb{Z}$-graded $k$-algebra domains to include:

$$k[X]_* = S_*/P$$

for homogeneous prime ideals $P \subset S_+$

the homogeneous coordinate rings of irreducible subsets of $\mathbb{P}^n_k$. These rings are:

- $\mathbb{Z}$-graded $k$-algebra integral domains, with $k[X]_0 = k$
- finitely generated in degree one by a basis $x_1, \ldots, x_n$ of $k[X]_1$.

We now construct a prevariety $(X, O_X)$ out of each such graded $k$-algebra $A_*$. The Set $X$ is the collection of maximal prime ideals $m_x \subset A_+$. The Topology is the Zariski topology, in which the algebraic sets:

$$X(I) = \{m_x \mid I \subset m_x\}$$

are the closed sets, for (radical) homogeneous ideals $I \subset A_+$. The Field of Rational Functions is:

$$k(X) = \left\{ \frac{F}{G} \mid F, G \in A_d \text{ and } G \neq 0 \right\} \subset k(A)$$

This is a subfield of $k(A)$. The elements of $k(X)$ are homogeneous of degree zero, which makes them (rational) functions on $X$. 

Concretely, a choice of basis \( x_0, ..., x_n \) of \( A_1 \) identifies \( A_\bullet = k[x_0, ..., x_n]/P \) and:

\[
\text{maxproj}(A_\bullet) = X = X(P) \subset \mathbb{P}^n_k
\]

This is an irreducible Zariski topological space by the Projective Nullstellensatz. For \( x = (a_0 : ... : a_n) \in X \), and \( \phi \in k(X) \),

\[
\phi(a_0, ..., a_n) = \frac{F(a_0, ..., a_n)}{G(a_0, ..., a_n)} = \frac{\lambda^d F(a_0, ..., a_n)}{\lambda^2 G(a_0, ..., a_n)} = \phi(\lambda a_0, ..., \lambda a_n)
\]

is well-defined, provided that \( G(a_0, ..., a_n) \neq 0 \). More abstractly,

**Definition 4.5.** A rational function \( \phi \in k(X) \) is regular at \( x \in X \) if

\[
\phi = \frac{F}{G} \quad \text{with} \quad G \notin m_x
\]

The rational functions that are regular at \( x \in X \) are elements of \( A_{(m_x)} \subset k(X) \), a local ring with residue field \( k \), in which the value \( \phi(x) \) is taken. The assignment:

\[
\mathcal{O}_X(U) = \{ \phi \in k(X) \mid \phi \text{ is regular at all points of } U \}
\]

defines the sheaf \( \mathcal{O}_X \) and the sheaved (Noetherian, irreducible) space \( \text{maxproj}(A_\bullet) \).

In contrast to Proposition 2.7, we have:

**Proposition 4.6.** \( \mathcal{O}_X(X) = k \) when \( (X, \mathcal{O}_X) = \text{maxproj}(A_\bullet) \).

**Proof.** Let \( \phi \in \mathcal{O}_X(X) \) and let \( I = \langle G \in A_d \mid G\phi \in A_d \rangle \) be the homogeneous ideal of denominators of \( I \). By assumption \( X(I) \) is empty, and if we could conclude (as in the affine case) that \( 1 \in I \), we’d have \( \phi \in A_0 = k \). Instead, we have:

\[
A_d \subset I \quad \text{for some } d \quad \text{(Lemma 4.4)}
\]

In other words, \( G\phi \in A_d \) for all \( G \in A_d \). This has the odd consequence that:

\[
G\phi^2 = (G\phi)\phi \in A_d, \quad G\phi^3 = (G\phi^2)\phi \in A_d, \text{etc}
\]

which gives an increasing chain of submodules:

\[
A_\bullet \subset A_\bullet + \phi A_\bullet \subset A_\bullet + \phi A_\bullet + \phi^2 A_\bullet \subset \cdots \subset G^{-1} A_\bullet
\]

of a principal graded \( A \)-module. Since \( A_\bullet \) is Noetherian, the chain stabilizes, and:

\[
\phi^n = f_0 + f_1 \phi + \cdots + f_{n-1} \phi^{n-1}
\]

for elements \( f_i \in A_\bullet \) in degree 0, this is an identity \( \phi^n = c_0 + c_1 \phi + \cdots + c_{n-1} \phi^{n-1} \) with coefficients in \( k = A_0 \), and then since \( k = \bar{k} \), it follows that \( \phi \in k, \) as desired. \( \square \)

So \( X \) isn’t affine (unless it is a point). But it is covered by affine varieties:

**Proposition 4.7.** Each sheaved space \( (X, \mathcal{O}_X) = \text{maxproj}(A_\bullet) \) is a prevariety.

**Proof.** Let \( G \in A_d \) be a non-zero element of positive degree \( d \). Then

\[
A_{(G)} = \left\{ \frac{F}{G^m} \mid \deg(F) = md \right\} \subset k(X)
\]

is a \( k \)-algebra domain, generated by \( y_i/G \), where \( y_i \) are a basis for \( A_d \). Moreover,

\[
k(A_{(G)}) = k(X)
\]

and \( (U_G, \mathcal{O}_X|_{U_G}) \) is isomorphic to \( \text{maxspec}(A_{(G)}) \), where \( U_G = X - X(G) \). In this case, we can conclude that \( G^m \) is in the ideal of denominators of each \( \phi \in \mathcal{O}_X(U_G) \) by the Projective Nullstellensatz, as in Proposition 2.7. \( \square \)
Example. The open cover of $\mathbb{P}_k^n$ by $n + 1$ affine spaces $U_0, ..., U_n$.

For each of the coordinate functions $x_0, ..., x_n \in k[x_0, ..., x_n]$, 

$$U_{x_i} = \text{maxspec}(k[x_0, ..., x_n](x_i)) = \text{maxspec}(k[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}])$$

is the affine $n$ space of points:

$$U_{x_i} = \{(a_0 : ... : a_n) \mid a_i \neq 0\} = \{(\frac{a_0}{a_i}, ..., 1, ..., \frac{a_n}{a_i})\}$$

Notice in passing that, $\text{PGL}(n, k)$ is an affine variety, by this Proposition.

A morphism from a prevariety $X$ to affine space $\mathbb{A}_k^n$ is given by regular functions:

$$g_1 = f^*(x_1), ..., g_n = f^*(x_n) \in \mathcal{O}_X(X)$$

via $f(x) = (g_1(x), ..., g_n(x))$. In particular, the only morphisms from a projective prevariety (or any prevariety with $\mathcal{O}_X(X) = k$) to $\mathbb{A}_k^n$ are the constant maps.

But what about morphisms from $X$ to $\mathbb{P}_k^n$? Is there a way to characterize these? The key is rational functions. Each prevariety $X$ has its rational function field:

$$k(X) = \lim \mathcal{O}_X(U)$$

When $X = \text{maxspec}(A)$ this is $k(A)$ and when $X = \text{maxproj}(A)$, it is $k(X)$. Moreover, if $U \subset X$ is any open subset, then $k(U) = k(X)$.

Definition 4.8. Rational functions $\phi_0, ..., \phi_n \in k(X)$ determine a rational map:

$$f : X \dashashrightarrow \mathbb{P}_k^n; f(x) = (\phi_0(x) : \cdots : \phi_n(x))$$

The domain of the rational map $f$ is larger than one might expect, since:

$$(\phi_0, ..., \phi_n) \text{ and } (\phi \cdot \phi_0, ..., \phi \cdot \phi_n)$$

determine the same rational map to $\mathbb{P}_k^n$ whenever $\phi \in k(X)^*$. This means that one may be able to expand the domain not just by different forms of $\phi_i = F_i/G_i$, but also by multiplying by convenient rational functions $\phi$.

Example. (a) The rational projection map $\pi : \mathbb{P}_k^2 \dashashrightarrow \mathbb{P}_k^1$ given by:

$$\left(\frac{x_1}{x_0} : \frac{x_2}{x_0}\right) = (1 : \frac{x_2}{x_1}) = (\frac{x_1}{x_2} : 1)$$

is well-defined on the open set $\mathbb{P}_k^2 - \{(1 : 0 : 0)\}$ but it cannot be extended further. When restricted to the projective line $X(x_1) \subset \mathbb{P}_k^2$, we get $\pi(a_0 : 0 : a_1) = (0 : 1)$ and when restricted to $X(x_2)$, we get $\pi(a_0 : a_1 : 0) = (0 : 1)$, so there is no way to give a value to $\pi(1 : 0 : 0)$ to extend $\pi$ to a continuous map. In fact, when restricted to each line through $(1 : 0 : 0)$, the projection map is a different constant.

(b) When $\pi$ is restricted to the conic $C = X(x_1^2 - x_0x_2) \subset \mathbb{P}_k^2$, however:

$$\pi|_C = (1 : \frac{x_2}{x_1}) = (\frac{x_1}{x_2} : 1) = (1 : \frac{x_1}{x_0})$$

with the last form of the map coming from the identity $x_2/x_1 = x_1/x_0$ in $k(C)$. Moreover, this rational map, defined everywhere, inverts $i : \mathbb{P}_k^1 \rightarrow C$ given by:

$$i = (1 : \frac{x_1}{x_0} : (\frac{x_1}{x_0})^2) = (\frac{x_0}{x_1}^2 : \frac{x_0}{x_1} : 1)$$

Proposition 4.9. A morphism $f : (X, \mathcal{O}_X) \rightarrow \mathbb{P}_k^n$ in the category of sheaved spaces is the same as a rational map that is defined at all points of $X$. 
Corollary 4.10. \( \mathbb{P}_k^1 \) and \( C \) from Example (b) above are isomorphic prevarieties. On the other hand, these two projective prevarieties come from the graded rings: 

\[ A_1 = \mathbb{k}[x_0, x_1] \quad \text{and} \quad A_{2*} = \mathbb{k}[x_0^2, x_0 x_1, x_1^2] \]

Exercise. \( \text{maxproj}(A_1) \) and \( \text{maxproj}(A_{2*}) \) are isomorphic varieties for all \( d > 0 \).

Proposition 4.11. Products of projective prevarieties are projective. 

Proof. It suffices to prove that \( \mathbb{P}_k^n \times \mathbb{P}_k^m \) is a projective prevariety, i.e. to locate this prevariety as a closed, irreducible subset of some \( \mathbb{P}_k^n \). Here it is:

\[ X = \{ \text{rank one } m \times n \text{ matrices} \} \subset \mathbb{P}_k^{(n+1)(m+1) - 1} \]

with projective coordinates \((a_{ij})\) for \( i = 0, ..., n \) and \( j = 0, ..., m \) and

\[ X = X(x_{ij} x_{kl} - x_{il} x_{kj}) \text{ (the vanishing of the two by two minors)} \]

Then \( X \) is set-theoretically equal to \( \mathbb{P}_k^n \times \mathbb{P}_k^m \) via the Segre embedding

\[ ((a_0 : ... : a_n), (b_0 : ... : b_m)) \mapsto (a_i b_j) \]

and the Cartesian projections are realized by restricting the rational projections:

\[ \pi^{\mathbb{P}_k^n} = (x_{i0} / x_{ij} : x_{20} / x_{ij} : ... : x_{n0} / x_{ij}) \quad \text{and} \quad \pi^{\mathbb{P}_k^m} = (x_{01} / x_{ij} : ... : x_{0m} / x_{ij}) \]

to \( X \) (for any choice of \( x_{ij} \)), where they are defined everywhere, hence morphisms. On each of the open affines \( U_i \times U_j = \mathbb{A}_k^n \times \mathbb{A}_k^m \), this agrees with the product of affine varieties, and so \((X, \pi^{\mathbb{P}_k^n}, \pi^{\mathbb{P}_k^m})\) is the universal triple. 

Corollary 4.12. Projective prevarieties are varieties. 

Proof. The diagonal in \( \mathbb{P}_k^n \times \mathbb{P}_k^n \) is the closed subset \( X(\{ x_{ij} - x_{ji} \}) \subset X \). It follows that quasi-projective prevarieties \( U \subset \text{maxproj}(A_1) \) are also varieties.

This choice of an arbitrary \( x_{ij} \) in the proof of Proposition 4.11 points to a useful way to think about morphisms from a projective variety \( X \) to \( \mathbb{P}_k^n \). If \( \phi_0, ..., \phi_n \) are rational functions defining a morphism \( \phi \), then we may choose \( G \in A_d \) for some (large) \( d \) so that \( G \phi_i = F_i \in A_d \) for all \( i \). We may then write \( f \) as:

\[ f(x) = (F_0(x) : ... : F_n(x)) \]

and although the values of each \( F_i(x) \) individually do not make sense, the ratio does give a well-defined point of projective space, provided that some \( F_i(x) \neq 0 \). Thus, from this point of view, the projection from \((1 : 0 : 0)\):

\[ \pi : \mathbb{P}_k^2 - - > \mathbb{P}_k^1 \]

can be written as \( \pi (x_0 : x_1 : x_2) = (x_1 : x_2) \)

and the isomorphism from \( \mathbb{P}_k^1 \) to the conic \( C \) can be written as:

\[ i : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^2; \ i(x_0 : x_1) = (x_0^2 : x_0 x_1 : x_1^2) \]
We finish this section with the “completion” of an affine variety. Let
\[ A = k[x_1, ..., x_n]/P \] with \( X = X(P) \subset \mathbb{A}^n_k \).

Then we may homogenize the ideal \( P \) by homogenizing its elements:
\[ P_{\text{hom}} = \langle f_{\text{hom}} = f(x_1/x_0, ..., x_n/x_0) \cdot x_0^d \mid f \in P, d = \deg(f) \rangle \subset k[x_0, ..., x_n] \]
into generators of \( P_{\text{hom}} \). This is a homogeneous prime ideal defining:
\[ Y = X(P_{\text{hom}}) \subset \mathbb{P}^n_k \]
satisfying \( Y \cap U_0 = X \).

This is the Zariski closure of \( Y_0 \cap X \subset U_0 \) as a subset of \( \mathbb{P}^n \). The main point is that this closure has an open cover by affine varieties \( Y_i = Y \cap U_i \) for all the other open affine space subsets \( U_i \subset \mathbb{P}^n \), allowing us to place each of the points in the closure of \( X \) in the interior of an open affine subvariety of \( Y \).

**Example.** By this prescription, the closure of the affine curve:
\[ X = X(x_2^2 - (x_1^3 + Ax_1 + B)) \subset \mathbb{A}^2_k \]
in the projective plane \( \mathbb{P}^2_k \) is:
\[ E = X(x_0x_2^2 - (x_1^3 + Ax_0x_2 + Bx_0^3)) \subset \mathbb{P}^2_k \]
which is obtained from \( X \) by adding the single point \((0 : 0 : 1) = E \cap X(x_0) \).

The two other affine spaces \( U_1, U_2 \subset \mathbb{P}^2_k \) intersect \( E \) in affine curves:
\[ X_1 = X(x_0x_2^2 - (1 + Ax_0x_2^2 + Bx_0^3)) \]
and \( X_2 = X(x_1 - (x_1^3 + Ax_2 + Bx_0^3)) \)
and it is in \( X_2 \) that we may study the elliptic curve “near” the extra point.

**Assignment 4.**

1. Prove that the projection: \( \pi(x_0 : ... : x_n) = (x_0 : ... : x_m) \) is not defined at the points of \( \Lambda = X((x_{m+1}, ..., x_n)) \). (a) Show that this is the case by finding:
\[ \pi^{-1}(a_0 : ... : a_m) \subset \mathbb{P}^n_k - \Lambda \] for each point \((a_0 : ... : a_m) \in \mathbb{P}^n_k \).

This is called the linear projection \( \pi_\Lambda : \mathbb{P}^n_k - \rightarrow \mathbb{P}^m_k \) from \( \Lambda \subset \mathbb{P}^n_k \).

(b) If \( Q = X(x_0x_3 - x_1x_2) \subset \mathbb{P}^3_k \), completely describe the projection:
\[ \pi_{(0:0:0:1)}|Q : Q -\rightarrow \mathbb{P}^2_k \]
Does it extend across \((0 : 0 : 0 : 1) \in X(Q)\)? (c) On the other hand, describe:
\[ \pi_\Lambda|Q : Q -\rightarrow \mathbb{P}^1_k \]
for \( \Lambda = \{(*:0:0)\} = X((x_2,x_3)) \) and show that this does extend across the points of \( \Lambda \) (as in Proposition 4.11.)

2. The \( d \)-uple embedding:
\[ f_d : \mathbb{P}^n_k \rightarrow \mathbb{P}^{(n+d)}_d \]
is given by \( f_d(x_0 : ... : x_n) = (... : x_I : ...) \) over all the multi-indices \( I \) of degree \( d \).

(a) If \( n = 1 \), the image of the \( d \)-uple embedding is the rational normal curve:
\[ C_d = \{(a_0^d : a_0^{d-1}a_1 : ... : a_1^d) \mid (a_0 : a_1) \in \mathbb{P}^1_k \} \]
corresponding to multi-indices \((d - i, i)\) generalizing the conic from Corollary 4.10.

Show that \( I(C_d) \) is generated by the \( 2 \times 2 \) minors of the matrix:
\[
\begin{bmatrix}
  x_{(d,0)} & x_{(d-1,1)} & \cdots & x_{(1,d-1)} \\
  x_{(d-1,1)} & x_{(d-2,2)} & \cdots & x_{(0,d)}
\end{bmatrix}
\]
(b) If \( d = 2 \), the embedding \( f_2 : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^{(n+2)−1} \) is the **Veronese embedding**. In this case, the monomials of degree 2 are all of the form \( x_i x_j \), and \( f_2 \) can be thought of as:

\[
f_2(a_0 : \ldots : a_n) = (... : a_i a_j : ...)
\]

whose coordinates can be arranged in a symmetric \( n + 1 \times n + 1 \) matrix \( A = (a_{i,j}) \). Show that the image is the rank one locus in symmetric all matrices \( (x_{i,j}) \), and is therefore cut out by the quadratic equations of the principal \( 2 \times 2 \) minors. Work out the explicit quadratic equations for the Veronese embedding of \( \mathbb{P}^2 \).

(c) In general, arrange the multi-indices in a convenient ordering to show that that \( d \)-uple embedding is an isomorphism from \( \mathbb{P}_k^n \) to its image via an appropriate inverse projective mapping.

3. The **Grassmannian** \( G(m, n) \) is the set of \( m \)-planes in \( k^n \) (e.g. \( G(1, n) = \mathbb{P}^n_k \)). Consider the rational map:

\[
\mathbb{P}({\text{Hom}}(k^m, k^n)) \dashrightarrow \mathbb{P}^{(n−1)}_m
\]
given by the \( m \times m \) **minors** of a matrix \( A \in \text{Hom}(k^m, k^n) \). Work out explicitly for the case \( m = 2 \) and \( n = 4 \) and convince yourself that the image is \( X(q) \subset \mathbb{P}^5_k \) for a suitable nonsingular (see Problem 5) quadratic polynomial. The image also can be interpreted as the set of indecomposable alternating tensors:

\[
v_1 \wedge \cdots \wedge v_m \text{ in } \wedge^m k^n
\]

4. (a) Prove Euler’s formula for homogeneous polynomials \( F \in k[x_0, \ldots, x_n]_d \).

\[
\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = dF
\]

(b) The projective tangent plane \( T_p(X(F)) \subset \mathbb{P}^n_k \) to \( X(F) \) at \( p \in X(F) \) is:

\[
\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}(p) = 0
\]

provided that the gradient \( \nabla(F)(p) \neq 0 \).

The affine tangent plane to \( X(f) \) for \( f \in k[x_1, \ldots, x_n] \) vanishing at \((0, \ldots, 0)\) is:

\[
X(f_1) \text{ where } f = f_1 + f_2 + \cdots + f_d \text{ are the homogeneous terms of } f
\]

Show that if \( F(p) = 0 \) and \( p = (1 : 0 : \ldots : 0) \), then:

\[
T_p(X(F)) \cap U_0 \text{ is the affine tangent plane to } X(f) = X(F) \cap U_0 \text{ at } (0, \ldots, 0)
\]

and that if \( \nabla(F)(p) = 0 \), then \( f_1 = 0 \) for the polynomial \( f = F(1, x_1/x_0, \ldots, x_n/x_0) \).

Thus, \( p \in X(F) \) is a singular point (no tangent plane) if and only if \( \nabla(F)(p) = 0 \).

In particular, if \( k = \mathbb{C} \) and \( \nabla(F)(p) \neq 0 \), then \( X(F) \) is a complex manifold of dimension \( n \) in a Zariski open neighborhood of \( p \in X(F) \).

(c) Show that the elliptic curve \( X(y^2 - x^3 - Ax - B) \) is non-singular at the “point at infinity” and find its projective tangent line.

5. In the projective plane \( \mathbb{P}^2_k \), the simplest singularities are simple nodes and cusps. If \( f(x_1, x_2) = f_2 + f_3 + \cdots + f_d \) is singular at \((0, 0)\), then:

\[
f_2(x_1, x_2) = (a_1 x_1 - a_2 x_2)(b_1 x_1 - b_2 x_2)
\]
(we’re assuming \( k = \mathbb{F} \)), and then:

(i) \( X(F) \) has a simple node at \((1 : 0 : 0)\) if \((a_2 : a_1) \neq (b_2 : b_1) \in \mathbb{P}^1\), i.e. if the linear factors of \( f_2 \) define different lines through \((0,0)\).

(ii) \( X(F) \) has a simple cusp at \((1 : 0 : 0)\) if the linear factors of \( f_2 \) are dependent (but not zero).

**Question.** How do we interpret this in terms of the tangent **cone**:

\[
\sum_{i,j} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} = 0
\]

at \( p \in X(F) \) of a singular point of \( X(F) \subset \mathbb{P}^2_k \)?

5. A homogeneous **quadric** is a quadratic form:

\[
q = \sum_{i \leq j} c_{i,j} x_i x_j \in k[x_0, ..., x_n]_2
\]

which is identified with the symmetric matrix:

\[
Q = \begin{bmatrix}
  c_{0,0} & \frac{1}{2} c_{0,1} & \cdots & \frac{1}{2} c_{0,n} \\
  \frac{1}{2} c_{0,1} & c_{1,1} & \cdots & \frac{1}{2} c_{1,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{2} c_{0,n} & \frac{1}{2} c_{1,n} & \cdots & c_{n,n}
\end{bmatrix}
\]

so that

\[
q(x_0, ..., x_n) = \vec{x}^T Q \vec{x}
\]

for the column vector \( \vec{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} \)

Prove that the singular locus of the **quadric hypersurface** \( X(q) \) is:

\[
\Lambda = \mathbb{P}(\ker(Q)) \subset \mathbb{P}^n_k
\]

so that in particular, \( X(q) \) is non-singular if and only if \( \det(Q) \neq 0 \).

Show (diagonalizing the quadric if like) that the projection from \( \Lambda \) realizes \( X(q) \) as the inverse image of a nonsingular quadric \( X(q_0) \) (closed up to include \( \Lambda \)) under the projection map:

\[
\pi_\Lambda : \mathbb{P}^n -\to \mathbb{P}(\text{im}(Q))
\]

This is called the **cone over the quadric** \( X(q_0) \subset \mathbb{P}(\text{im}(Q)) \).

6. Prove that the only automorphisms of \( \mathbb{P}^n_k \) (as projective varieties) are the natural (transitive) action of \( \text{PGL}(n,k) \). What are the automorphisms of a non-singular quadric \( Q \subset \mathbb{P}^n_k \)?