Abstract Algebra. Math 6310. Bertram/Utah 2022-23.

Introducing Tensors and Complexes

We start with the tensor product. Let R be an object of $\mathcal{C}Ring$.

Definition. A product $\cdot : A \times A \to A$ on an R-module A is R-*bilinear* if:

$$(ra_1 + sa_2) \cdot a = ra_1 a + sa_2 a \text{ and } a \cdot (ra_1 + sa_2) = raa_1 + saa_2$$

for all $r, s \in R$ and $a, a_1, a_2 \in A$. An R-module A with this product is an R-*algebra*.

Note. R-algebras need not be commutative (or even associative!) to be interesting (see e.g. Lie algebras). In this, our commutative phase, we will restrict ourselves to the category $\mathcal{C}Alg_R$ of *commutative* (and associative) R-algebras with 1, in which $f : A \to B$ is an R-module homomorphism that is also a ring homomorphism. In this context, the data of a commutative R-algebra is equivalent to that of a commutative ring A with 1 equipped with a ring homomorphism:

$$f : R \to A \text{ with } r \cdot a = f(r) \cdot a \text{ as the } R\text{-module structure}$$

(and vice versa, given an R-algebra A with 1, we let $f(r) = r \cdot 1$ define $f : R \to A$).

Definition. A is generated *as an R-algebra* by elements $\{a_\lambda\} \in A$ for $\lambda \in \Lambda$ if the (finite) *monomials* $\prod_{i=0} a_{\lambda_i}^{d_i} \in A$ in the a_λ generate A *as an R-module*.

In particular, A is *finitely* generated (as an R-algebra) if and only if there is a surjective morphism of R-algebras:

$$R[x_0, ..., x_n] \to A;\ x_i \mapsto a_i \text{ for some } n$$

from the free polynomial algebra in a finite number of variables to A.

Examples. (a) The polynomial algebras $A = R[x_0, ..., x_n]$ are the *free* R-algebras. They are also graded (by degree in the x_i) and each graded summand:

$$R[x_0, ..., x_n]_d \text{ is a free and finitely generated } R\text{-module}$$

though overall, A is not finitely generated as an R-module!

(b) R itself is the initial object of the category $\mathcal{C}Alg_R$.

(c) $\mathcal{C}Alg_{\mathbb{Z}}$ is the category of commutative rings with 1.

(d) Products (as commutative rings/R-modules) are products (as R-algebras).

Gorenstein Algebras. A finitely generated, graded k-algebra (for a field k)

$$A_\bullet = \bigoplus_{i=0} A_i \text{ with } A_0 = k \text{ is } \textit{Gorenstein} \text{ if}$$

- $A_\bullet$ is finitely generated as a k-module (i.e. as a vector space over A_0),
- the nonzero summand A_d of largest degree is one-dimensional, and:
- for each $i = 0, ..., d$, the k-bilinear maps:

$$A_i \times A_{d-i} \to A_d = k \text{ are perfect pairings}$$

i.e. if $0 \neq a \in A_i$ then $ab \neq 0$ for some $b \in A_{d-i}$ (and vice versa).

It follows that A_i and A_{d-i} are dual vector spaces, and the Hilbert function:

$$h_A(i) = \dim(A_i) \text{ is palindromic}$$

i.e. h_A reads the same right (of zero) and left (of d).

Examples (of Gorenstein algebras). The Hilbert function of a Gorenstein algebra with $d = 1$ reads as [1 1], and there is only one, isomorphic to the k-algebra $k[x]/x^2$.

A Gorenstein algebra with $d = 2$ has Hilbert function [1 n 1] and such an algebra is given by a (non-degenerate) quadratic form:

$$q : V \times V \to k$$

for a vector space $V = A_1$ of dimension n.

From the perfect pairing of Poincaré duality, one concludes that the even degree part of the graded cohomology algebra of a compact oriented manifold M of even dimension is a Gorenstein algebra. For example, the cohomology algebra of complex projective space $\mathbb{CP}^n$ is isomorphic to $\mathbb{R}[x]/x^{n+1}$ where x is assigned degree **two**. (To include the odd degree part of the cohomology algebra, one needs to upgrade the Gorenstein algebra to a *super* (or commutative-graded) algebra).

Definition. The *tensor product* of R-modules M and N is the R-module $M \otimes_R N$ that is: generated as an R-module by the symbols $m \otimes n$ for $m \in M, n \in N$ with two kinds of relations:

(i) "scalar swapping" relations $r(m \otimes n) = rm \otimes n = m \otimes rn$ (for all r, m, n)

and the "bilinear" relations (for all r, s, m_i, n_j)

(ii) $(rm_1 + sm_2) \otimes n = r(m_1 \otimes n) + s(m_2 \otimes n)$ and
$m \otimes (rn_1 + sn_2) = r(m \otimes rn_1) + s(m \otimes n_2)$

Remark. Despite its name, the tensor product is neither a product nor a coproduct in the category of R-modules, since we've already seen that $\oplus$ serves both roles. However, the map:

$$\otimes : M \times N \to M \otimes_R N;\ (m, n) \mapsto m \otimes n$$

is R-**bi**linear (by construction!) and satisfies the following universal property:

UT. Every R-bilinear map to an R-module:

$$b : M \times N \to P$$

factors through a unique R-module homomorphism $\bar{b} : M \otimes_R N \to P$.

Note. A bilinear map is not an R-module homomorphism since, for example, $b(m, 0) = 0$ and $b(0, n) = 0$, but $b(m, n) \neq 0$ for some m, n (unless $b = 0$).

Examples. (a) $R \otimes_R M = M$, and $R^n \otimes_R M = M^n$ for free modules R^n.

(b) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}$ for $d = gcd(m, n)$,which is in contrast with

the primary decomposition $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/e\mathbb{Z}$ for $e = lcm(m, n)$. More generally, if R/I and R/J are cyclic R-modules, then

$$R/I \otimes R/J \cong R/(I + J) \text{ is also cyclic, generated by } 1 \otimes 1$$

(c) It is important to realize that b is not usually surjective (the image of b is the set of *indecomposable* tensors). For example, if $V = k^n$ and $W = k^m$ are vector spaces over k, the general element of $V \otimes_k W$ has the form:

$$\sum_{i,j} c_{i,j} e_i \otimes f_j \text{ where the } e_i \text{ and } f_j \text{ are basis vectors of } V, W$$

and cannot be compressed any further. That is, the $e_i \otimes f_j$ are a basis of the k-vector space $V \otimes_k W$ (as one could also have deduced from (a)).

However, in the category of commutative R-algebras, matters are different:

Proposition 1. The tensor product is the coproduct in the category $\mathcal{C}Alg_R$.

Proof. First, we have to establish the tensor product of A and B is an R-algebra. The multiplication will be:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$$

extended by R-linearity to the free R-module generated by $a_i \otimes b_j$. It is an exercise to check that this respects the relations, i.e. is a well-defined product on $A \otimes_R B$, and then associativity and commutativity are inherited from A and B. Note that:

$$1 = 1 \otimes 1 \text{ and } 0 = 0 \otimes 0 = 0 \otimes b = a \otimes 0 \text{ for any } a, b$$

and more generally, the ring homomorphism $f : R \to A \otimes_R B$ is given by:

$$f(r) = (r \cdot 1) \otimes 1 = 1 \otimes (r \cdot 1)$$

The two *injection* morphisms defining the tensor as a coproduct are:

$$i : A \to A \otimes_R B;\ i(a) = a \otimes 1 \text{ and } j : B \to A \otimes_R B;\ j(b) = 1 \otimes b$$

If C is an object of $\mathcal{C}Alg_R$ with morphisms $f : A \to C$ and $g : B \to C$, then:

$$h(a \otimes b) = f(a) \cdot g(b)$$

is the desired morphism, uniquely determined by $h \circ i = f$ and $h \circ j = g$. $\square$

Let $\mathcal{A}$ be an abelian category.

Definition. A **chain complex** $C_\bullet$ of objects of $\mathcal{A}$) is a sequence of morphisms:

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots$$

with the property that $d_i \circ d_{i+1} = 0$ for all i.

From our toolkit, we conclude that there is a unique *monomorphism*

$$\mathrm{im}(d_{i+1}) \xrightarrow{h_i} \ker(d_i)$$

factoring the image monomorphism $\iota_{i+1} : I_{i+1} \to C_i$ (of d_{i+1}) through the kernel monomorphism $k_i : K_i \to C_i$ (of d_i) by the universal property of the kernel. Then:

$$H_i(C_\bullet) := \mathrm{coker}(h_i) \text{ is the } \textbf{homology of the complex } C_\bullet \text{ in degree } i$$

and the chain complex $C_\bullet$ is **exact** at C_i if $H_i(C_\bullet) = 0$, i.e. if h_i is an isomorphism.

Remark. A chain complex is more accurately notated as $(C_\bullet, d_\bullet)$, including the data of the *differentials* d_i, but we will usually be sloppy and leave them out.

Definition. A morphism $f_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ of chain complexes is a collection:

$$f_i : C_i \to C'_i$$

of morphisms that commute with the differentials, i.e. such that:

$$\begin{array}{ccccccc}
\cdots & \to & C_i & \xrightarrow{d_i} & C_{i-1} & \to & \cdots \\
 & & f_i \downarrow & & f_{i-1} \downarrow & & \\
\cdots & \to & C'_i & \xrightarrow{d'_i} & C'_{i-1} & \to & \cdots
\end{array}$$

is a *commuting diagram* (i.e. $f_{i-1} \circ d_i = d'_{i-1} \circ f_i$)

Exercise. The chain complexes (with chain morphisms) are an abelian category, which we will denoted by $\mathcal{C}h_{\mathcal{A}}$. The key point is to show that the kernels:

$$\cdots \to \ker(f_i) \to \ker(f_{i-1}) \to \cdots$$

and cokernels:

$$\cdots \to \mathrm{coker}(f_{i+1}) \to \mathrm{coker}(f_i) \to \cdots$$

of the f_i morphisms form complexes (with induced differentials). Try this yourself, or else see the proof of the Snake Lemma below.

The exact complexes (aka *exact sequences*) in $\mathcal{C}h_{\mathcal{A}}$ are of particular importance.

Examples. (i) Every object C of $\mathcal{A}$ may be "complexified" into a one-term complex:

$$\cdots \to 0 \to C = C_i \to 0 \to \cdots$$

for a fixed i (with zero differentials). Morphisms of such complexes are morphisms as objects of $\mathcal{A}$. The only such complex that is *exact* is the zero complex.

(ii) The homology of a two-term (mini) complex:

$$\cdots \to 0 \to C_i \stackrel{d_i}{\to} C_{i-1} \to 0 \to \cdots$$

is $H_i(C_\bullet) = \ker(d_i)$ and $H_{i-1}(d_i) = \mathrm{coker}(d_{i-1})$ so the complex is exact at C_i if and only if d_i is a monomorphism, exact at C_{i-1} if and only if d_{i-1} is an epimorphism, and exact if and only if d_i is both mono and epi, i.e. an isomorphism.

Short Complexes. A short (three term) complex:

$$0 \to C_{i+1} \to C_i \to C_{i-1} \to 0 \text{ is}$$

(a) *Right exact* if it is exact at C_i and C_{i-1} and

(b) *Left exact* if it is exact at C_{i-1} and C_i, and so

it is exact if and only if it is both left and right exact.

Remarks. (a) A short complex with $\mathrm{H}_i(C_\bullet) = 0$ (exact in the middle) is written:

$$C_{i+1} \to C_i \to C_{i-1}$$

removing the zeroes to indicate that it may fail to be exact at the ends. Likewise:

$$C_{i+1} \to C_i \to C_{i-1} \to 0 \text{ and } 0 \to C_{i+1} \to C_i \to C_{i-1}$$

denote right and left exact short complexes, respectively.

(b) For $\mathcal{A} = \mathcal{M}od_R$, there are distinguished short exact sequences:

$$S \subset C_i \stackrel{q}{\to} C_i/S$$

Each short exact sequence is isomorphic to one of these via the following diagram:

$$\begin{array}{ccccccc} 0 & \to & C_{i+1} & \stackrel{d_{i+1}}{\to} & C_i & \stackrel{d_i}{\to} & C_{i-1} \to 0 \\ & & \downarrow & & || & & \uparrow \\ & & \mathrm{im}(d_{i+1}) & \subset & C_i & \stackrel{q}{\to} & C_i/\mathrm{im}(d_{i+1}) \end{array}$$

in which all the vertical arrows are isomorphisms. For this reason, and because a morphism of R-modules is a (R-linear) mapping of sets, we will retreat from the generality of an abstract abelian category and focus on the categories of R-modules.

We finish this section with some diagram chasing.

The Snake Lemma. A morphism from a right exact to a left exact short complex:

$$\begin{array}{ccccccc} & & A & \stackrel{f}{\to} & B & \stackrel{g}{\to} & C & \to & 0 \\ & & a\downarrow & & b\downarrow & & c\downarrow & & \\ 0 & \to & A' & \stackrel{f'}{\to} & B' & \stackrel{g'}{\to} & C' & & \end{array}$$

of R-modules induces a (long) exact complex:

$$\ker(a) \stackrel{\overline{f}}{\to} \ker(b) \stackrel{\overline{g}}{\to} \ker(c) \stackrel{\delta}{\to} \mathrm{coker}(a) \stackrel{\overline{f'}}{\to} \mathrm{coker}(b) \stackrel{\overline{g'}}{\to} \mathrm{coker}(c)$$

(i.e. it is exact at all the middle terms).

Proof. First, let's establish what the morphisms are:

(a) The first two morphisms are defined as follows:

$\overline{f}(\alpha) := f(\alpha) \in \ker(b)$ if $\alpha \in \ker(a)$ since $b \circ f(\alpha) = f' \circ a(\alpha) = 0$ and

$\overline{g}(\beta) := g(\beta) \in \ker(c)$ if $\beta \in \ker(b)$ since $c \circ g(\beta) = g' \circ b(\beta) = 0$

and it is immediate from the definition that $\overline{g} \circ \overline{f} = 0$.

(b) The last two morphisms also use:

$$f' \circ a(\alpha) = b \circ f(\alpha) \text{ and } g' \circ b(\beta) = c \circ g(\beta)$$

to conclude that $f'(\mathrm{im}(a)) \subset \mathrm{im}(b)$ and $g'(\mathrm{im}(b)) \subset \mathrm{im}(c)$ and then:

$$\overline{f'}(\alpha' + \mathrm{im}(a)) := f'(\alpha') + \mathrm{im}(b) \text{ and } \overline{g'}(\beta' + \mathrm{im}(b)) := g'(\beta') + \mathrm{im}(c)$$

are well-defined, and again it is immediate that $\overline{g'} \circ \overline{f'} = 0$.

(c) δ is the snake morphism. Given $\gamma \in \ker(c)$, choose $\beta \in B$ such that

$$g(\beta) = \gamma$$

and then note that $g' \circ b(\beta) = c(\gamma) = 0$ so there is a (unique) $\alpha' \in A'$ such that

$$f'(\alpha') = b(\beta)$$

Finally, let

$$\delta(\gamma) := \alpha' + \mathrm{im(a)} \in \mathrm{coker}(a)$$

The only ambiguity was in the choice of β. But if $\beta_0 \in \ker(g)$, then

$$g(\beta_0) = 0 \text{ and so } \beta_0 = f(\alpha_0) \text{ and } b(\beta_0) = f' \circ a(\alpha_0)$$

Thus if we replace β with $\beta + \beta_0$, then we replace:

$$\alpha' + \mathrm{im}(a) \text{ with } \alpha' + a(\alpha_0) + \ \mathrm{im}(a) \text{ which are the same cosets}$$

(d) $\delta \circ \overline{g} = 0$. If $\gamma = g(\beta)$ and $b(\beta) = 0$, then $f'(\alpha') = 0$ and $\alpha' = 0$.

(e) $\overline{f'} \circ \delta = 0$. If $\alpha' + \mathrm{im}(a) = \delta(\gamma)$, then $f'(\alpha') = b(\beta) \in \ \mathrm{im}(b)$.

Thus the sequence of the Lemma is a complex. Exactness is left as an exercise. □

Remark. If the sequences are both *exact* in the lemma, then the full long complex:

$$0 \to \ker(a) \stackrel{\overline{f}}{\to} \ker(b) \stackrel{\overline{g}}{\to} \ker(c) \stackrel{\delta}{\to} \mathrm{coker}(a) \stackrel{\overline{f'}}{\to} \mathrm{coker}(b) \stackrel{\overline{g'}}{\to} \mathrm{coker}(c) \to 0$$

is exact (i.e. $\ker(a) \to \ker(b)$ is injective and $\mathrm{coker}(b') \to \mathrm{coker}(c')$ is surjective). Thus, the snake morphism can be seen as an "exact" link between the (left exact) sequence of kernels, and the (right exact) sequence of cokernels in this context.

Example. For natural numbers m, n, consider the morphism:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \stackrel{mn}{\to} & \mathbb{Z} & \to & \mathbb{Z}/mn\mathbb{Z} & \to & 0 \\ & & m\downarrow & & || & & \downarrow & & \\ 0 & \to & \mathbb{Z} & \stackrel{n}{\to} & \mathbb{Z} & \to & \mathbb{Z}/n\mathbb{Z} & \to & 0 \end{array}$$

Then the snake lemma gives the isomorphism: $\delta : \ker(\mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}) \stackrel{\sim}{\to} \mathbb{Z}/m\mathbb{Z}$ (all other kernels and cokernels are zero).

Corollary. In the the snake lemma, if a and c are isomorphisms then b is, too.

Proof. Tracing through the long exact sequence, we find:

$$\ker(a) \to \ker(b) \to \ker(c) \quad \text{and} \quad \mathrm{coker}(a) \to \mathrm{coker}(b) \to \mathrm{coker}(c)$$

so in fact more is true:

- If a and c are monomorphisms, then b is a monomorphism, and
- If a and c are epimorphisms, then b is an epimorphism. □

Non-Example. It is useful to know limitations implied by this Corollary. Note that:

$$0 \to \mathbb{Z}/2\mathbb{Z} \stackrel{f}{\to} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \stackrel{g}{\to} \mathbb{Z}/2\mathbb{Z} \to 0$$

is a short exact sequence, with $f(x) = (x, 0)$ and $g(x, y) = y$. But

$$0 \to \mathbb{Z}/2\mathbb{Z} \stackrel{f'}{\to} \mathbb{Z}/4\mathbb{Z} \stackrel{g'}{\to} \mathbb{Z}/2\mathbb{Z} \to 0$$

is also exact, with $f'(x) = 2x$ and $g'(y) = y \pmod 2$. There is **no way** to fill in:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 \\ & & || & & b\downarrow & & || & & \\ 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathbb{Z}/4\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 \end{array}$$

with b to get a morphism of complexes since the two groups are *not isomorphic.*

The Corollary above is a special case of the more general:

Five Lemma. Given a morphism of exact (in the middle) complexes of R-modules:

$$\begin{array}{ccccccccc} A & \stackrel{f}{\to} & B & \stackrel{g}{\to} & C & \stackrel{h}{\to} & D & \stackrel{i}{\to} & E \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ A' & \stackrel{f'}{\to} & B' & \stackrel{g'}{\to} & C' & \stackrel{h'}{\to} & D' & \stackrel{i'}{\to} & E' \end{array}$$

if b, d are isomorphisms, a is surjective and e is injective, then c is an isomorphism.

Proof. Suppose $\gamma' \in C'$. Then $h'(\gamma') = d(\delta)$ for some δ since d is an epimorphism and then $e \circ i(\delta) = i' \circ d(\delta) = i' \circ h'(\gamma') = 0$, so $i(\delta) = 0$ since e is injective, and:

$$\delta = h(\gamma) \text{ for some } \gamma \in C \text{ by exactness}$$

Now consider $\gamma' - c(\gamma) \in C'$. Then $h'(\gamma' - c(\gamma)) = d(\delta) - d \circ h(\gamma) = 0$, so:

$$\gamma' - c(\gamma) = g'(\beta') \text{ for some } \beta' \in B' \text{ by exactness}$$

and we may choose $\beta \in B$ so that $b(\beta) = \beta'$ since b is an epimorphism. Thus:

$$c(\gamma + g(\beta)) = c(\gamma) + c \circ g(\beta) = c(\gamma) + g'(\beta') = \gamma'$$

and c is surjective! Summarizing: b, d surjective $+\ e$ injective $\Rightarrow$ c is surjective
Similarly, one shows b, d injective $+\ a$ surjective $\Rightarrow c$ is injective. □

Remark. The two parts of the proof may be split off as a pair of "four" lemmas.