University of Utah, Department of Mathematics
August 2017, Algebra Qualifying Exam

Show all your work, and provide reasonable justification for your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Determine, up to isomorphism, the groups of order 18.

2. How many elements of S_7 commute with the permutation $(123)(45)(67)$?

3. Determine all prime ideals of the ring $\mathbb{R}[x, ix]$, where x is an indeterminate over $\mathbb{R}$, and $i^2 = -1$.

4. Let $R = \mathbb{Z}[i]$ and $S = \mathbb{Z}[\sqrt{-3}]$. Does there exist a ring homomorphism $R \longrightarrow S$? Does there exist a ring homomorphism $S \longrightarrow R$?

5. Determine, up to conjugacy, all *real* 5×5 matrices with characteristic polynomial $x(x^2+1)^2$.

6. Let n be a positive integer, V a $\mathbb{C}$-vector space of rank n, and $T \colon V \longrightarrow V$ a $\mathbb{C}$-linear map with the property that the for each $c \in \mathbb{C}$ the eigen space $\{v \in V \mid T(v) = cv\}$ has rank at most 1. Prove that there exists a $w \in V$ such that the set $\{w, T(w), \ldots, T^{n-1}(w)\}$ is a basis for V.

7. Prove that every real matrix that commutes with the matrix
$$A := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
is of the form $rI + sA$, for some real numbers r, s.

8. Determine the minimal polynomial of π over the field $\mathbb{Q}((\pi^2+1)/(\pi-1))$.

   (Use, without proof, that π is transcendental!)

9. Let $\alpha = 2^{1/5} + 5^{1/2}$. Determine the extension degree of α over $\mathbb{Q}$.

10. Determine the minimal polynomial of $\sqrt{4+\sqrt{7}}$ over $\mathbb{Q}$. Determine the Galois group of this polynomial.