Holomorphic maps between Riemann surfaces and Normal Families

Holomorphic maps between tori and other Riemann surfaces.

1. Construct a surjective holomorphic map $\mathbb{D} \to \mathbb{C}$.

2. Recall annuli $A(D) = \{D < |z| < 1\}$ for $0 \leq D < 1$. For which D is there a holomorphic map $f : A(D) \to A(D)$ of degree 2, meaning that the induced homomorphism $f_* : \pi_1(A(D)) \to \pi_1(A(D))$ is multiplication by 2?

3. Let $T_\Lambda = \mathbb{C}/\Lambda$ and $T_\Gamma = \mathbb{C}/\Gamma$ be two tori with $\Lambda, \Gamma \subset \mathbb{C}$ two lattices. Prove that every holomorphic map $f : T_\Lambda \to T_\Gamma$ is either constant or a covering map. Hint: Consider two cases. If the image of $f_* : \pi_1(\Lambda) \to \pi_1(\Gamma)$ has infinite index, show that f is constant by lifting to suitable covering spaces so that you get a map from a compact to a noncompact Riemann surface. If the image has finite index, show that the lift $\tilde{f} : \mathbb{C} \to \mathbb{C}$ between universal covers grows linearly: $|\tilde{f}(z)| \leq A|z| + B$ for some constants A, B, and conclude that $\tilde{f}$ is a linear function.

4. Let $\Lambda = \mathbb{Z}[i]$ be the Gaussian integers, and $\Gamma = \mathbb{Z}[\omega]$ be the Eisenstein integers, where $\omega = e^{\pi i/3}$. Show that every holomorphic map $T_\Lambda \to T_\Gamma$ and $T_\Gamma \to T_\Lambda$ is constant. Hint: This should reduce to showing that there is no $a \neq 0$ so that $a\Lambda \subset \Gamma$ or $a\Gamma \subset \Lambda$. Specifically, show that $\omega \notin \mathbb{Q}(i)$ and $i \notin \mathbb{Q}(\omega)$ (fields obtained by adjoining a number).

5. Recall that the Schwarz lemma says that a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is either a hyperbolic isometry or a strict contraction: $\|df(v)\| < \|v\|$ for all nonzero tangent vectors v, where $\|\cdot\|$ is hyperbolic norm. Show that this cannot be improved to the statement: f is either an isometry or there is some $c < 1$ such that $\|df(v)\| \leq c\|v\|$. Hint: $z \mapsto z^2$.

Normal families and Montel's theorem.

Below, $\Omega \subset \mathbb{C}$ will be a connected open set. Recall that a family $\mathcal{F}$ of functions $\Omega \to \mathbb{C}$ is *locally bounded* if every $z \in \Omega$ has a neighborhood U and there is some C such that $|f(w)| < C$ for every $w \in U$ and every $f \in \mathcal{F}$.

6. Prove the converse of Montel's theorem: if a sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ converges uniformly on compact sets to a function $f : \Omega \to \mathbb{C}$, then the sequence is locally bounded.

7. Prove Vitali's theorem: Let $\Omega \subseteq \mathbb{C}$ be a domain, let $f_n : \Omega \to \mathbb{C}$ be a locally bounded sequence of holomorphic functions, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let

$$A = \{z \in \Omega \mid \lim_{n\to\infty} f_n(z) \text{ exists and } f(z) = \lim_{n\to\infty} f_n(z)\}$$

   If A has a limit point in Ω, then $f_n \to f$ uniformly on compact sets.

Limits of holomorphic functions.

Here we will collect some properties of limits of sequences of functions. Let $\Omega \subseteq \mathbb{C}$ be a connected open set (domain), and let $f_n : \Omega \to \mathbb{C}$ be holomorphic functions converging uniformly on compact sets to a function $f : \Omega \to \mathbb{C}$.

8. The function f is holomorphic. Hint: Goursat plus Morera, but make sure you verify all assumptions.

9. $f_n' \to f'$ uniformly on compact sets, and likewise for higher derivatives. Hint: Cauchy.

10. If $f_n(z) \neq 0$ for all n and all $z \in \Omega$, then either $f(z) \neq 0$ for all $z \in \Omega$, or $f \equiv 0$. Hint: Rouché.

11. (Hurwitz) If all f_n are injective, then either f is injective or f is constant.