Flows and Lie derivative

Integral curves and flows

1. Let M be a manifold and V, W two smooth vector fields on M. Suppose that V and W are equal outside a compact set. Also suppose that V generates a global flow (i.e. integral curves for V are defined on all of $\mathbb{R}$). Show that W also generates a global flow.

2. Let $f : M \to N$ be a smooth map, V, W vector fields on M, N respectively. We say that V, W are *f-related* if for every $x \in M$ we have $df(V(x)) = W(f(x))$. Assuming V and W are f-related prove that if $\alpha : (a, b) \to M$ is an integral curve for V then $f \circ \alpha$ is an integral curve for W.

3. Let M be a compact manifold without boundary, $f : M \to \mathbb{R}$ smooth. Suppose that all points in $[a, b] \subset \mathbb{R}$ are regular values for f. Show that $f^{-1}(-\infty, a]$ and $f^{-1}(-\infty, b]$ are diffeomorphic. Hint: This is part of the Morse theorem. Use Problem 2. and construct a vector field on M so that the flow generated by it takes $f^{-1}(-\infty, a]$ to $f^{-1}(-\infty, b]$. The vector field on M doesn't have to be f-related to a vector field on $\mathbb{R}$ everywhere, only on $f^{-1}[a, b]$, and choosing a Riemannian metric on M can be useful.

Lie derivative

Let $M = \mathbb{R}^2$, $X = x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$

4. Let $\omega = y\ dx + x\ dy$. Compute the Lie derivative $L_X(\omega)$.

5. Let $Y = xy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Compute the Lie derivative $L_X Y$.

Identities

There are many identities between Lie derivatives, Lie brackets of vector fields, exterior derivative and interior product $\lrcorner$. You will recall from homework 8 that if ω is a k-form and V a vector field then $V \lrcorner \omega$ is a $(k-1)$-form defined by $V \lrcorner \omega(X_1, \cdots, X_{k-1}) = \omega(V, X_1, \cdots, X_{k-1})$. It is convenient to denote by i_X the operator $i_X\omega = X \lrcorner \omega$. Examples of identities include $L_X Y = [X, Y]$, Cartan's Magic Formula $L_X = di_X + i_X d$ and the identity $d\omega(X, Y) = \frac{1}{2}(X\omega(Y) - Y\omega(X) - \omega([X, Y]))$ for vector fields X, Y and a

1-form ω from an earlier homework. Identities are local and to prove them we can work in a chart.

A good strategy that simplifies the calculations is to use the fact that d, i_X, L_X satisfy the Leibnitz rule and you can try to use this to show that if some identity involving a form ω is satisfied for forms ω_1, ω_2 then it is satisfied for $\omega = \omega_1 \wedge \omega_2$. This reduces the problem to the case when ω is a 0-form or $\omega = dx_i$. You can also usually assume that one of the vector fields X or Y in the identity is $\partial/\partial x_1$ (and also 0 but that case is usually obvious). I think I didn't lose any factorials in the identities, but if you think I did let me know.

6. The goal of this Problem and the next is to prove the identity

$$[L_X, i_Y] = i_{[X,Y]}$$

   That is, for any form ω,

$$L_X(i_Y(\omega)) - i_Y(L_X(\omega)) = i_{[X,Y]}(\omega)$$

   Here show as suggested above that if this is true for ω_1, ω_2 then it's true for $\omega_1 \wedge \omega_2$. They also say "both sides are derivations", in this case of degree -1 (the degree of a form goes down by 1).

7. Finish the proof of the identity in the previous problem. You can assume $Y = \partial/\partial x_1$, $\omega = dx_i$ and $X = \sum_j a_j \frac{\partial}{\partial x_j}$, where a_j are functions.

8. If X, Y are vector fields and ω is a 1-form then

$$(L_X\omega)(Y) = X(\omega(Y)) - \omega([X, Y])$$

   Hint: Here the Leibnitz rule involves only $f\omega$, i.e. show that if the identity holds for ω then it holds for $f\omega$, and then the calculation is reduced to $\omega = dx_i$.