Frobenius theorem

1. Let $X = \frac{\partial}{\partial x} + f\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y} + g\frac{\partial}{\partial z}$ be two vector fields on $\mathbb{R}^3$ and $\Delta = span\{X, Y\}$. Show that Δ is integrable iff

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} + f\frac{\partial g}{\partial z} - g\frac{\partial f}{\partial z} = 0$$

Miscellaneous

2. Find an example of a complete metric space X (for example, $\mathbb{R}$) and a function $f : X \to X$ such that $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$, but f does not have any fixed points.

3. Let X, Y be two vector fields on a manifold M. Show that

$$L_{[X,Y]} = [L_X, L_Y]$$

   where $L_{[X,Y]}$ is the Lie derivative of vector fields with respect to the Lie bracket $[X, Y]$ and $[L_X, L_Y]$ is the commutator $L_X L_Y - L_Y L_X$. Hint: Recall the Jacobi identity.

4. The formula $L_{[X,Y]} = [L_X, L_Y]$ is also true when L is viewed as an operator on forms, but the proof is harder. Follow the usual strategy:

   (a) Show that if $L_{[X,Y]}\omega_i = [L_X, L_Y](\omega_i)$ for $i = 1, 2$ then $L_{[X,Y]}\omega = [L_X, L_Y](\omega)$ for $\omega = \omega_1 \wedge \omega_2$. This is formal, using the Leibnitz rule.

   (b) Show that $L_{[X,Y]}\omega = [L_X, L_Y](\omega)$ when ω is a 0-form (i.e. a function).

   (c) Show that $L_{[X,Y]}\omega = [L_X, L_Y](\omega)$ when $\omega = dx_i$ (in local coordinates). You can use Cartan's Magic Formula, which will be particularly simple here since one of terms will be 0.

5. Let X, Y be two vector fields on a manifold M and assume $X(p) = Y(p) = 0$ for some $p \in M$. Show that $L_X Y(p) = 0$.

6. Let X_t for $t \in \mathbb{R}$ be a 1-parameter family of vector fields on $\mathbb{R}^n$. By this I mean that the function $(x, t) \mapsto X_t(x) \in \mathbb{R}^n$ is a smooth function $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. We also refer to X_t as a *time dependent vector field.* A curve $c : (a, b) \to \mathbb{R}^n$ is an integral curve of X_t provided $c'(t) =$

$X_t(c(t))$. The following trick reduces the existence problem to the existence of integral curves for the usual vector fields. Define a vector field Y on $\mathbb{R}^n \times \mathbb{R}$ by $Y(x,t) = X_t(x) + \frac{\partial}{\partial t}$. Thus the $\mathbb{R}$-component is always $\frac{\partial}{\partial t}$ and the $\mathbb{R}^n$-component at level t is given by X_t. Let α be an integral curve of Y. Show that the projection of α to $\mathbb{R}^n$ is an integral curve of X_t. This implies existence and uniquencess theorems about equations of the form $y' = F(y,t)$.

7. In a similar way, higher order ODE's can be reduced to integrating vector fields. The usual ODE giving a vector field has the form $y' = F(y)$ and then we use F to define the vector field. For example, suppose we have an equation $y'' = F(y, y')$. Show that this can be reduced to integrating a vector field on $\mathbb{R}^n \times \mathbb{R}^n$ by introducing a new variable z that represent y' and the equation is

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} z \\ F(y,z) \end{pmatrix}$$

In general, differential equations of the form $y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$ can be reduced to integrating vector fields.