Tangent spaces, immersions, submersions

Most problems here are from Guillemin-Pollack.

Tangent spaces

1. Find the tanget space, as a subspace of $\mathbb{R}^3$, of the paraboloid defined by
$$x^2 + y^2 - z^2 = a$$
at $(\sqrt{a}, 0, 0)$ where $a > 0$.

2. Find the tangent space, as a subspace of the space of complex $n \times n$ matrices, of the unitary group $U(n)$ at $I \in U(n)$.

Immersions and local diffeomorphisms

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a local diffeomorphism. Prove that the image of f is an open interval and f maps $\mathbb{R}$ diffeomorphically onto this interval. Hint: The set of points x where $f'(x) > 0$ $[f'(x) < 0]$ is open.

4. Construct a local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not injective.

5. Suppose $f : X \to Y$ is an injective local diffeomorphism. Show that f is a diffeomorphism onto its image.

6. Let $G : \mathbb{R}^2 \to S^1 \times S^1$ be defined by $G(s,t) = (e^{2\pi is}, e^{2\pi it})$. It's a local diffeomorphism from the plane to the torus. Show that if L is an irrational slope line given by the equation $t = as$ with a irrational, then G is injective on L.

7. Let $x_1, x_2, \cdots, x_n$ be the standard coordinate functions on $\mathbb{R}^n$, i.e. x_i is the projection to the i^{th} coordinate. Let $X \subset \mathbb{R}^n$ be a k-submanifold. Show that for every $x \in X$ there are some k coordinate functions $x_{i_1}, \cdots, x_{i_k}$ that form a local coordinate system around x. In other words, the projection to these k coordinates is a chart around x.

8. (Inverse Function Theorem generalized) Let $f : X \to Y$ be a smooth map, $Z \subset X$ a submanifold, and assume that f is injective on Z and it is a local diffeomorphism at every point $z \in Z$. Show that f maps a neighborhood of Z diffeomorphically onto its image. Hint: Use Problem 5. Correction: This is **false** as stated, e.g. the image of Z

may accumulate on itsef. To make the statement correct assume also that $f|Z : Z \to Y$ is a proper map. Hint: Start with an exhaustion K_i of Y, let $A_i = K_i \smallsetminus int(K_{i-1})$, let $Z_i = f^{-1}(A_i)$ and let U_i be a neighborhood of Z_i on which f is injective and so that $\overline{f(U_i)} \cap A_j = \varnothing$ if $|i-j| > 1$. Then $U = \cup U_i$ almost works (f may not be injective on the union of 3 consecutive U_i). You can shring U_i's so that images of U_i and U_j are disjoint if $|i-j| > 1$, and then shring further so that f is injective on $U_i \cup U_{i+1}$.

Submersions

9. If X is compact (and nonempty!) and Y connected, then a submersion $f : X \to Y$ is surjective. Hint: Submersions send open sets to open sets.

10. Let P be a degree $m > 0$ homogeneous polynomial in k variables, meaning that

$$P(tx_1, \cdots, tx_k) = t^m P(x_1, \cdots, x_k)$$

Show that any $a \neq 0$ is a regular value of $P : \mathbb{R}^k \to \mathbb{R}$, so that $\{P(x) = a\}$ is a submanifold of $\mathbb{R}^k$.

11. (Stack of records theorem) Let $f : X \to Y$ be smooth with X compact and $\dim X = \dim Y$. Let y be a regular value of f. Show that there is a neighborhood U of y such that $f^{-1}(U)$ is a finite disjoint union $V_1 \sqcup \cdots \sqcup V_N$ of open sets such that f maps each V_i diffeomorphically to U. See the picture and hint in Guillemin-Pollack.

12. Prove that the set of 2×2 matrices of rank 1 is a 3-dimensional submanifold of $M(2) = \mathbb{R}^4$. Hint: Consider $\det : M(2) \smallsetminus \{0\} \to \mathbb{R}$.

Miscellaneous

13. Consider the set $\mathbb{C}^{n+1} \smallsetminus \{0\}$ with the equivalence relation

$$(X_0, X_1, \cdots, X_n) \sim (Y_0, Y_1, \cdots, Y_n)$$

if there is a nonzero complex number λ such that $X_j = \lambda Y_j$ for all j. The quotient space is the *complex projective space* $\mathbb{C}P^n$. The equivalence class of $(X_0, X_1, \cdots, X_n)$ is usually denoted $[X_0 : X_1 : \cdots : X_n]$.

Show that this is a manifold of dimension $2n$. More specifically, let $U_j = \{[X_0, X_1, \cdots, X_n] \mid X_j \neq 0\}$ and define $\phi_j : U_j \to \mathbb{C}^n$ by

$$[X_0, X_1, \cdots, X_n] \mapsto \left(\frac{X_0}{X_j}, \frac{X_1}{X_j}, \cdots, \frac{X_{j-1}}{X_j}, \frac{X_{j+1}}{X_j}, \cdots, \frac{X_n}{X_j} \right)$$

Show that these define an atlas.

14. The *real projective space* $\mathbb{R}P^n$ is defined similarly, but the equivalence relation is on $\mathbb{R}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Show similarly that $\mathbb{R}P^n$ is a manifold of dimension n. Also show that $\mathbb{R}P^1$ is diffeomorphic to S^1. (It is also true that $\mathbb{C}P^1$ is diffeomorphic to S^2.)

15. Show that $SL_n(\mathbb{R})$ is path connected. Hint: From linear algebra, every matrix in $SL_n(\mathbb{R})$ can be transformed to the identity I by elementary row operations. Show that one can interpolate a path between any two consecutive such matrices. For example, adding the first row to the second can be interpolated by adding t times the first row for $t \in [0, 1]$.

It is also true, by a similar argument, that $GL_n(\mathbb{R})$ has two components, but $GL_n(\mathbb{C})$ is connected.