Mostly a review

Most of these are from Lee's book.

Tangent spaces

1. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by

$$f(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$

Show that $(0, 1)$ is a regular value of f and compute the tangent space of $f^{-1}(0, 1)$ at $(0, 0, 0, 1)$ as a subspace of $\mathbb{R}^4$.

2. If X, Y are manifolds then $X \times Y$ is a manifold (you don't have to prove this). Let $(p, q) \in X \times Y$. Show that $T_{(p,q)}(X \times Y)$ is the direct sum $T_pX \oplus T_qY$ where we identify T_pX with the subspace

$$Im(d\phi_p : T_pX \to T_{(p,q)}(X \times Y))$$

and $\phi : X \to X \times Y$ is inclusion $x \mapsto (x, q)$, and similarly for T_qY. Hint: It is also useful to consider projections $X \times Y \to X$ and $X \times Y \to Y$.

Abstract approaches

3. For a smooth manifold X we denote by $C^\infty(X)$ the algebra of smooth functions $X \to \mathbb{R}$ and by $C^0(X)$ the algebra of continuous functions $X \to \mathbb{R}$. Thus $C^\infty(X) \subset C^0(X)$ is a subalgebra. When $f : X \to Y$ is a continuous map there is an induced "pull-back" $f^* : C^0(Y) \to C^0(X)$ defined by $\phi \mapsto \phi f$. Show that f is smooth iff $f^*(C^\infty(Y)) \subseteq C^\infty(X)$.

4. For a manifold X and $p \in X$ denote by $\mathcal{D}_pX$ the vector space of derivations at p. This exercise shows how to define the derivative via derivations (thought of as tangent vectors) directly without referring to charts.

Let $f : X \to Y$ be a smooth map. Define

$$df_p : \mathcal{D}_pX \to \mathcal{D}_{f(p)}Y$$

by

$$df_p(D)(g) = D(gf)$$

(thus for $D \in \mathcal{D}_pX$ this formula defines $df_p(D) \in \mathcal{D}_{f(p)}$).

(i) Verify the chain rule from this definition.

(ii) For the case $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, $p = 0$, $f(p) = 0$ and the identifications $\mathcal{D}_0\mathbb{R}^n = \mathbb{R}^n$ from class (i.e. $v \in \mathbb{R}^n$ corresponds to the derivation ∂_v) and likewise for $\mathbb{R}^m$, show that df_0 agrees with the usual definition of the derivative.

Lie groups

5. Show that Lie groups $O(n)$ and $U(n)$ are compact for $n \geqslant 1$.

6. Show that Lie groups $GL_n(\mathbb{R})$ for $n \geqslant 1$, $SL_n(\mathbb{R})$ for $n \geqslant 2$, $Sp_{2n}(\mathbb{R})$ for $n \geqslant 1$ are not compact.

7. Let G be a Lie group, $m : G \times G \to G$ the multiplication, and $i : G \to G$ the inversion $g \mapsto g^{-1}$. We will identify $T_I(G \times G)$ with $T_I(G) \oplus T_I(G)$ as in Problem 2..

   (i) Show that the derivative of m at the identity I is
   $$dm_I : T_I(G \times G) = T_I(G) \oplus T_I(G), (u, v) \mapsto u + v$$
   Hint: Restrict m to $G \times \{I\}$ and $\{I\} \times G$.

   (ii) Show that the derivative of i at the identity is
   $$di_I : T_I(G) \to T_I(G), u \mapsto -u$$
   Hint: Consider $G \to G \times G$, $g \mapsto (g, g^{-1})$ and compose with m.

8. Let G be a connected Lie group and U any neighborhood of the identity. Show that every $g \in G$ is a product of finitely many elements of U (i.e. U generates G). Hint: the subgroup generated by G is open.

9. Let J be the diagonal $(n+1) \times (n+1)$ matrix whose bottom right entry is -1 and all other diagonal entries are 1. The group $O(n, 1)$ consists of $(n + 1) \times (n + 1)$ real matrices M such that $MJM^\intercal = J$. This is similar to the definition of orthogonal groups, except that instead of preserving the standard inner product the matrices must preserve the pairing $(x_i) \cdot (y_i) = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$. Show that $O(n, 1)$ is a Lie group and that
   $$T_I(O(n, 1)) = \{A \in M(n + 1) \mid AJ + JA^\intercal = 0$$
   The group $O(n, 1)$ is essentially (up to some finite index subgroups) the group of isometries of the hyperbolic n-space. The group $O(3, 1)$ is the *Lorentz group* that stars in general relativity.

10. Show that $O(n, 1)$ is not compact, e.g. start with $O(1, 1)$.