Intersection and Degree Theory

Degree Theory

In this section all manifolds are compact, connected, oriented and have the same dimension n.

1. If $f : X \to Y$ and $g : Y \to Z$ are smooth, show that $\deg(gf) = \deg(g)\deg(f)$. Hint: First find some $z \in Z$ which is a regular value for g and so that every point in $g^{-1}(z)$ is a regular value for f.

2. Let $f, g : S^n \to S^n$ be two smooth maps such that $f(x) \neq -g(x)$ for every $x \in S^n$. Prove that $\deg f = \deg g$.

3. Show that for every smooth map $f : S^{2k} \to S^{2k}$ there exists a point $x \in S^{2k}$ such that either $f(x) = x$ or $f(x) = -x$.

Degree Theory for continuous maps

4. Prove the Boundary Theorem for continuous maps: if W is compact oriented with boundary, Y is compact oriented connected, $f : W \to Y$ is continuous, then $\deg \partial f : \partial W \to Y = 0$.

5. Prove that there is no continuous retraction of any compact manifold to its boundary. You can assume that both the manifold and its boundary are connected.

Sections of bundles and self-intersection of the 0-section

6. Let $\pi : E \to B$ be a vector bundle and for convenience assume that B is compact. Show that there are finitely many smooth sections $\sigma_1, \cdots, \sigma_N : B \to E$ such that for every $b \in B$ the vectors $\sigma_1(b), \cdots, \sigma_N(b)$ span the vector space $\pi^{-1}(b)$. Hint: Working in a chart, show that this is possible in a neighborhood of every point.

7. Let $\pi : E \to B$ be a vector bundle and for convenience assume that B is compact. Show that there is a section $\sigma : B \to E$ which is transverse to the submanifold of E consisting of 0's in each fiber (i.e. the 0-section). Moreover, every section can be approximated by a section transverse to the 0-section. Hint: Use Problem 6 and the transversality theorem.

8. Let $\pi : E \to B$ be a vector bundle and for convenience assume that B is compact. Also asume that it's an n-dimensional bundle and that the base B is an n-manifold (with the same n). Let $\sigma : B \to E$ be a section transverse to the 0-section Z, and assume B and the bundle are oriented (so in particular Z is transversally oriented). Show that $I(\sigma, Z)$ is independent of the choice of σ. This is called the *Euler number* of the bundle. It also equals $I(Z, Z)$ (so it's 0 when the dimension is odd).

9. Show that every odd dimensional oriented vector bundle admits an orientation reversing automorphism. Hint: $v \mapsto -v$.

Miscellaneous

10. Let X be a manifold, and identify it with the diagonal $\Delta \subset X \times X$ via $x \mapsto (x, x)$. Prove that the normal bundle of Δ in $X \times X$ is isomorphic to the tangent bundle TX. Hint: $(v, -v) \leftarrow\!\shortmid v$.

11. Compute the self-intersection mod 2 of $\mathbb{R}P^n \subset \mathbb{R}P^{2n}$. Here, a point in $\mathbb{R}P^{2n}$ is given in homogeneous coordinates as $[x_0 : x_1 : \cdots : x_{2n}]$ and the submanifold $\mathbb{R}P^n$ is defined as the set of such points where $x_{n+1} = \cdots = x_{2n} = 0$. Hint: Explicitly isotope $\mathbb{R}P^n$ so it is transverse to the initial copy. E.g. when $n = 1$ you can move the point $[a : b : 0]$ to $[a : b : b]$ via $[a : b : tb]$, then to $[a : 0 : b]$. Now the intersection is only $[1 : 0 : 0]$. Comment: this also works in $\mathbb{C}P^{2n}$ and there you can do the oriented intersection number.