Exterior derivative and the Stokes theorem

Exterior derivative

1. Define the following $(n-1)$-form ω on $\mathbb{R}^n \setminus \{0\}$:

$$\omega = \sum_{k=1}^{n} (-1)^k \frac{x_k \; dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n}{(x_1^2 + x_2^2 + \cdots + x_n^2)^{n/2}}$$

   Note that the denominator is $\|x\|^n$.

   (i) Show that ω is invariant under scaling, i.e. if $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is defined by $f(x) = \lambda x$ for $\lambda \neq 0$, then $f^*(\omega) = \omega$.

   (ii) Show that ω is a closed form, i.e. $d\omega = 0$.

2. Let ω be a 1-form on a manifold M and X, Y two smooth vector fields on M. As a check that you understand all definitions, show that

$$2 \; d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

   Here, e.g. $\omega(Y)$ is the smooth function on M obtained by plugging in the vector $Y(p)$ into ω for every $p \in M$ and $X\omega(Y)$ is the derivative of this function along X. The bracket $[X, Y]$ is the Lie bracket. This identity can be used to define the exterior derivative of a 1-form without local coordinates. There are analogous and more complicated formulas for higher order forms, see Spivak. You should write it out in coordinates and for simplicity you may assume that $\dim M = 2$. Thus $\omega = f \; dx + g \; dy$ and by linearity you can assume e.g. $\omega = f \; dx$.

Stokes Theorem and calculus

3. Show that the Stokes theorem for a closed interval $[a, b]$ is just the Fundamental Theorem of Calculus.

4. Derive Green's formula in the plane from the Stokes theorem: If $W \subset \mathbb{R}^2$ is a compact surface with boundary $\gamma = \partial W$ and $f, g : \mathbb{R}^2 \to \mathbb{R}$ are smooth, then

$$\int_\gamma f \; dx + g \; dy = \int_W \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \; dy$$

5. Derive the Divergence theorem from the Stokes theorem: Let $W \subset \mathbb{R}^3$ be a compact 3-manifold and let $\vec{n}$ be the outward normal to ∂W. If $\vec{F} = (f_1, f_2, f_3)$ is a vector field in $\mathbb{R}^3$ then

$$\int_W (div\vec{F}) dx\ dy\ dz = \int_{\partial W} (\vec{n} \cdot \vec{F})\ dA$$

where dA is the area form on ∂A.

6. Derive the (classical) Stokes theorem from the Stokes theorem(!): Let $S \subset \mathbb{R}^3$ be a compact orented surface with boundary and $\vec{F} = (f_1, f_2, f_3)$ a vector field in $\mathbb{R}^3$. Then

$$\int_S (curl\vec{F} \cdot n) dA = \int_{\partial S} f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

where $\vec{n}$ is a unit normal to S compatible with orientations and dA is the area form.

More Stokes

7. Let W be a compact oriented manifold with boundary $\partial W = X$, and let $n = \dim X$. If $f : W \to Y$ is a smooth map and $\omega \in \Omega^n(Y)$ a closed form then $\int_X f^*(\omega) = 0$.

8. Let $f, g : X \to Y$ be smooth homotopic maps with X compact, oriented, and $\partial X =$. Let $n = \dim X$, and $\omega \in \Omega^n(Y)$ a closed form. Show that $\int_X f^*(\omega) = \int_X g^*(\omega)$.

9. Let M be a compact oriented n-manifold without boundary. Let $\omega \in \Omega^n(M)$, so ω is closed. Show that if ω is exact then $\int_M \omega = 0$. More generally, show that if ω is an exact p-form on a manifold X and $M \subset X$ is an oriented compact p-manifold without boundary then $\int_M \omega = 0$. Normally, one shows that a form is not exact by computing an integral like this and showing it is not 0.

10. Show that the form from Problem 1. is not exact. Hint: By Problem 9, it suffices to show that $\int_{S^{n-1}} \omega \neq 0$. This might be a painful calculation, but here is a trick. The form $\eta = \sum_k (-1)^k x_k dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n$ is defined on all of $\mathbb{R}^n$, agrees with ω on S^{n-1} and $d\eta$ is a multiple of the volume form on $\mathbb{R}^n$, so apply Stokes on the unit ball.

11. Show that the form ω (or equivalently η) from Problem 10 is the volume form on S^{n-1}. This is of course harder than Problem 10. Hint: One approach is this. First check that this is true at the north pole $(0, 0, \cdots, 0, 1)$. Then show that ω (and η) are invariant under rotations in 2 coordinates, i.e. under the maps such as

$$(x_1, x_2, \cdots, x_n) \mapsto (x_1 \cos\alpha + x_2 \sin\alpha, -x_1 \sin\alpha + x_2 \cos\alpha, \cdots, x_n)$$

and finally show that any point on the sphere can be rotated to the north pole by composing such 2-coordinate rotations.