Notes on Sela's work: Limit groups and Makanin-Razborov diagrams

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Abstract

This is the first in a series of papers giving an alternate approach to Zlil Sela's work on the Tarski problems. The present paper is an exposition of work of Kharlampovich-Myasnikov and Sela giving a parametrization of $Hom(G,\mathbb{F})$ where G is a finitely generated group and $\mathbb{F}$ is a non-abelian free group.

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1 The Main Theorem

This is the first of a series of papers giving an alternative approach to Zlil Sela's work on the Tarski problems [31, 30, 32, 24, 25, 26, 27, 28]. The present paper is an exposition of the following result of Kharlampovich-Myasnikov [9, 10] and Sela [30]:

Theorem. *Let G be a finitely generated non-free group. There is a finite collection $\{q_i : G \to \Gamma_i\}$ of proper quotients of G such that, for any homomorphism f from G to a free group F, there is $\alpha \in Aut(G)$ such that $f\alpha$ factors through some q_i.*

A more precise statement is given in the Main Theorem. Our approach, though similar to Sela's, differs in several aspects: notably a different measure of complexity and a more geometric proof which avoids the use of the full Rips theory for finitely generated groups acting on $\mathbb{R}$-trees, see Section 7. We attempted to include enough background material to make the paper self-contained.

Notation 1.1. $\mathbb{F}$ is a fixed non-abelian free group. Finitely generated (finitely presented) is abbreviated fg (respectively fp).

The main goal of [30] is to give an answer to the following:

Question 1. *Let G be an fg group. Describe the set of all homomorphisms from G to $\mathbb{F}$.*

Example 1.2. When G is a free group, we can identify $Hom(G, \mathbb{F})$ with the cartesian product $\mathbb{F}^n$ where $n = rank(G)$.

Example 1.3. If $G = \mathbb{Z}^n$, let $\mu : \mathbb{Z}^n \to \mathbb{Z}$ be the projection to one of the coordinates. If $h : \mathbb{Z}^n \to \mathbb{F}$ is a homomorphism, there is an automorphism $\alpha : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $h\alpha$ factors through μ. This provides an explicit (although not 1-1) parametrization of $Hom(G, \mathbb{F})$ by $\mathbb{F} \times Hom(G, \mathbb{Z})$.

Example 1.4. When G is the fundamental group of a closed genus g orientable surface, let $\mu : G \to F_g$ denote the homomorphism to a free group of rank g induced by the (obvious) retraction of the surface to the rank g graph. It is a folk theorem[1] that for every homomorphism $f : G \to \mathbb{F}$ there is an automorphism $\alpha : G \to G$ (induced by a homeomorphism of the surface) so that $f\alpha$ factors through μ. The theorem was generalized to the case when G is

[1] see [35, 33]

the fundamental group of a non-orientable closed surface by Grigorchuk and Kurchanov [7]. Interestingly, in this generality the single map μ is replaced by a finite collection $\{\mu_1, \cdots, \mu_k\}$ of maps from G to a free group F. In other words, for all $f \in Hom(G, \mathbb{F})$ there is $\alpha \in Aut(G)$ induced by a homeomorphism of the surface such that $f\alpha$ factors through some μ_i.

Another goal is to understand the class of groups that naturally appear in the answer to the above question, these are called limit groups.

Definition 1.5. Let G be an fg group. A sequence $\{f_i\}$ in $Hom(G, \mathbb{F})$ is *stable* if, for all $g \in G$, the sequence $\{f_i(g)\}$ is eventually always 1 or eventually never 1. The *stable kernel* of $\{f_i\}$, denoted $\underrightarrow{Ker}\ f_i$, is

$$\{g \in G \mid f_i(g) = 1 \text{ for almost all } i\}.$$

An fg group Γ is a *limit group* if there is an fg group G and a stable sequence $\{f_i\}$ in $Hom(G, \mathbb{F})$ so that $\Gamma \cong G/\underrightarrow{Ker}\ f_i$.

Remark 1.6. One can view each f_i as inducing an action of G on the Cayley graph of $\mathbb{F}$, and then can pass to a limiting $\mathbb{R}$-tree action (after a subsequence). If the limiting tree is not a line, then $\underrightarrow{Ker}\ f_i$ is precisely the kernel of this action and so Γ acts faithfully. This explains the name.

Definition 1.7. An fg group Γ is *residually free* if for every element $\gamma \in \Gamma$ there is $f \in Hom(\Gamma, \mathbb{F})$ such that $f(\gamma) \neq 1$. It is *ω-residually free* if for every finite subset $X \subset \Gamma$ there is $f \in Hom(\Gamma, \mathbb{F})$ such that $f|X$ is injective.

Exercise 2. *Free groups and free abelian groups are ω-residually free.*

Exercise 3. *The fundamental group of $n\mathbb{P}^2$ for $n = 1$, 2, or 3 is not ω-residually free, see [13].*

Exercise 4. *Every ω-residually free group is a limit group.*

Exercise 5. *An fg subgroup of an ω-residually free group is ω-residually free.*

Exercise 6. *Every non-trivial abelian subgroup of an ω-residually free group is contained in a unique maximal abelian subgroup. For example, $F \times \mathbb{Z}$ is not ω-residually free for any non-abelian F.*

Lemma 1.8. *Let $G_1 \to G_2 \to \cdots$ be an infinite sequence of epimorphisms between fg groups. Then the sequence*

$$Hom(G_1, \mathbb{F}) \leftarrow Hom(G_2, \mathbb{F}) \leftarrow \cdots$$

eventually stabilizes (consists of bijections).

Proof. Embed $\mathbb{F}$ as a subgroup of $SL_2(\mathbb{R})$. That the corresponding sequence of varieties $Hom(G_i, SL_2(\mathbb{R}))$ stabilizes follows from algebraic geometry, and this proves the lemma. □

Corollary 1.9. *A sequence of epimorphisms between* ($\omega-$)*residually free groups eventually stabilizes.*

Lemma 1.10. *Every limit group is* ω*-residually free.*

Proof. Let Γ be a limit group, and let G and $\{f_i\}$ be as in the definition. Without loss, G is fp. Now consider the sequence of quotients

$$G \to G_1 \to G_2 \to \cdots \to \Gamma$$

obtained by adjoining one relation at a time. If Γ is fp the sequence terminates, and in general it is infinite. Let $G' = G_j$ be such that $Hom(G', \mathbb{F}) = Hom(G, \mathbb{F})$. All but finitely many f_i factor through G' since each added relation is sent to 1 by almost all f_i. It follows that these f_i factor through Γ and each non-trivial element of Γ is sent to 1 by only finitely many f_i. By definition, Γ is ω-residually free. □

Definition 1.11. A GAD[2] of a group G is a finite graph of groups decomposition of G with abelian edge groups in which some of the vertices are designated QH[3] and some others are designated *abelian*, and the following holds.

- A QH-vertex group is the fundamental group of a compact surface S with boundary and the boundary components correspond to the incident edge groups (they are all infinite cyclic). Further, S carries a pseudoAnosov homeomorphism (so S is a torus with 1 boundary component of $\chi(S) \leq -2$).

- An abelian vertex group A is abelian (!). The subgroup $P(A)$ of A generated by incident edge groups is the *peripheral subgroup.* $\overline{P}(A)$ is the subgroup of A that dies under every homomorphism from A to $\mathbb{Z}$ that kills $P(A)$, i.e.

$$\overline{P}(A) = \cap\{Ker(f) \mid f \in Hom(A, \mathbb{Z}), P(A) \subset Ker(f)\}.$$

[2]Generalized Abelian Decomposition
[3]Quadratically Hanging

The non-abelian non-QH vertices are *rigid*.

Remark 1.12. If Δ is a GAD for a fg group G, and if A is an abelian vertex group of Δ, then there is an epimorphism $G \to A/\overline{P}(A)$. Hence, $A/\overline{P}(A)$ is fg. Since it is also torsion free, $A/\overline{P}(A)$ is free. It follows that $A = A_0 \oplus \overline{P}(A)$ with A_0 a retract of G.

Definition 1.13. The *modular group* $Mod(\Delta)$ associated to a GAD Δ of G is the subgroup of $Aut(G)$ generated by

- inner automorphisms of G,
- Dehn twists[4] in elements of G that centralize an edge group of Δ,
- unimodular[5] automorphisms of abelian vertex groups that are identity on peripheral subgroups and all other vertex groups, and
- automorphisms induced by homeomorphisms of surfaces S underlying QH-vertices that fix all boundary components. If S is closed and orientable, we require the homeomorphisms to be orientation-preserving[6].

The *modular group of* G, denoted $Mod(G)$, is the subgroup of $Aut(G)$ generated by $Mod(\Delta)$ for all GAD's Δ of G. At times it will be convenient to view $Mod(G)$ as a subgroup of $Out(G)$. In particular, we will say that an element of $Mod(G)$ is *trivial* if it is an inner automorphism.

Definition 1.14. We define a hierarchy of fg groups – if a group belongs to this hierarchy it is called a CLG[7].

Level 0 of the hierarchy consists of fg free groups.

A group Γ belongs to level $\leq n+1$ iff either it has a free product decomposition $\Gamma = \Gamma_1 * \Gamma_2$ with Γ_1 and Γ_2 of level $\leq n$ or it has a homomorphism $\rho : \Gamma \to \Gamma'$ with Γ' of level $\leq n$ and it has a GAD such that

- ρ is injective on the peripheral subgroup of each abelian vertex group.
- ρ is injective on each edge group E and at least one of the images of E in a vertex group of the one-edged splitting induced by E is a maximal abelian subgroup.

[4]See Section 3 for a definition.
[5]The induced automorphism of $A/\overline{P}(A)$ has determinant 1.
[6]We will want our homeomorphisms to be products of Dehn twists.
[7]Constructible Limit Group

- The image of each QH-vertex group is a non-abelian subgroup of H.
- For every rigid vertex group B, ρ is injective on the "envelope" $\tilde{B}$ of B, defined by first replacing each abelian vertex with the peripheral subgroup and then letting $\tilde{B}$ be the subgroup of the resulting group generated by B and by the centralizers of incident edge-groups.

Example 1.15. A free abelian group of rank n is a CLG of level $n-1$. The fundamental group of a closed surface S with $\chi(S) \leq -2$ is a CLG of level 1. For example, an orientable genus 2 surface is a union of 2 punctured tori and the retraction to one of them determines ρ. Similarly, a non-orientable genus 2 surface is the union of 2 punctured Klein bottles.

Example 1.16. Start with the circle and attach to it 3 surfaces with one boundary component, with genera 1, 2, and 3 say. There is a retraction to the surface of genus 3 that is the union of the attached surfaces of genus 1 and 2. This retraction sends the genus 3 attached surface say to the genus 2 attached surface by "pinching a handle". The GAD has a central vertex labeled $\mathbb{Z}$ and there are 3 edges that emanate from it, also labeled $\mathbb{Z}$. Their other endpoints are QH-vertex groups. The map induced by retraction satisfies the requirements so the fundamental group of the 2-complex built is a CLG.

Example 1.17. Choose a primitive[8] w in the fg free group F and form $\Gamma = F *_{\mathbb{Z}} F$, the double of F along $\langle w \rangle$ (so $1 \in \mathbb{Z}$ is identified with w on both sides). There is a retraction $\Gamma \to F$ that satisfies the requirements (both vertices are rigid), so Γ is a CLG.

The following can be proved by induction on levels.

Exercise 7. *Every CLG is fp, in fact coherent. Every fg subgroup of a CLG is a CLG. (Hint: a graph of coherent groups over fg abelian groups is coherent.)*

Exercise 8. *Every abelian subgroup of a CLG Γ is fg and free, and there is a uniform bound to the rank. There is a finite $K(\Gamma, 1)$.*

Exercise 9. *Every non-abelian, freely indecomposable CLG admits a "principal splitting" over $\mathbb{Z}$: $A *_{\mathbb{Z}} B$ or $A *_{\mathbb{Z}}$ with A, B non-cyclic, and in the latter case $\mathbb{Z}$ is maximal abelian in the whole group.*

Exercise 10. *Every CLG is ω-residually free.*

[8] no proper root

The last exercise is more difficult than the others. It explains where the conditions in the definition of CLG come from. The idea is to construct homomorphisms $G \to \mathbb{F}$ by choosing complicated modular automorphisms of G, composing with ρ and then with a homomorphism to $\mathbb{F}$ that comes from the inductive assumption.

Example 1.18. Consider an index 2 subgroup H of an fg free group F and choose $g \in F \setminus H$. Suppose that $G := H *_{\langle g^2 \rangle} \langle g \rangle$ is freely indecomposable and admits no principal cyclic splitting. There is the obvious map $G \to F$, but G is not a limit group (Exercise 9 and Theorem 1.25). This shows the necessity of the last condition in the definition of CLG's. [9]

Definition 1.19. A *factor set* for a group G is a finite collection of proper quotients $\{q_i : G \to G_i\}$ such that if $f \in Hom(G, \mathbb{F})$ then there is $\alpha \in Mod(G)$ such that $f\alpha$ factors through some q_i.

Main Theorem ([9, 10, 31]). *Let G be an fg group that is not free. Then, G has a factor set $\{q_i : G \to \Gamma_i\}$ with each Γ_i a limit group. If G is not a limit group, we can always take $\alpha = Id$.*

Corollary 1.20. *Iterating the construction of the Main Theorem (for Γ_i's etc.) yields a finite tree of groups terminating in groups that are free.*

Proof. If $\Gamma \to \Gamma'$ is a proper epimorphism between limit groups, then since limit groups are residually free, $Hom(\Gamma', \mathbb{F}) \subsetneq Hom(\Gamma, \mathbb{F})$. We are done by Lemma 1.8. ∎

Definition 1.21. The tree of groups and epimorphisms provided by Corollary 1.20 is called an *MR-diagram*[10] for G (with respect to $\mathbb{F}$). If

$$G \xrightarrow{q} \Gamma_1 \xrightarrow{q_1} \Gamma_2 \xrightarrow{q_2} \cdots \xrightarrow{q_{m-1}} \Gamma_m$$

is a branch of an MR-diagram and if $f \in Hom(G, \mathbb{F})$ then we say that f *MR-factors* through this branch if there are $\alpha \in Mod(G)$ (which is Id if G is not a limit group), $\alpha_i \in Mod(\Gamma_i)$, for $1 \leq i < m$, and $f_m \in Hom(\Gamma_m, \mathbb{F})$ (recall Γ_m is free) such that $f = f_m q_{m-1} \alpha_{m-1} \cdots q_1 \alpha_1 q \alpha$.

[9] The element $g := a^2 b^2 a^{-2} b^{-1} \in H := \langle a, b^2, bab^{-1} \rangle \subset F := \langle a, b \rangle$ is such an example. This can be seen from the fact that if $\langle x, y, z \rangle$ denotes the displayed basis for H, then $g^2 = x^2 y x^{-2} y^{-1} z^2 y z^{-2}$ is Whitehead reduced and each basis element occurs at least 3 times.

[10] for Makanin-Razborov, cf. [14, 15, 19].

Remark 1.22. The key property of an MR-diagram for G is that, for $f \in Hom(G, \mathbb{F})$, there is a branch of the diagram through which f MR-factors. This provides an answer to Question 1 in that $Hom(G, \mathbb{F})$ is parametrized by branches of an MR-diagram and, for each branch as above, $Mod(G) \times Mod(\Gamma_1) \times \cdots \times Mod(\Gamma_{m-1}) \times Hom(\Gamma_m, \mathbb{F})$. Note that if Γ_m has rank n, then $Hom(\Gamma_m, \mathbb{F}) \cong \mathbb{F}^n$.

In [28], Sela constructed an MR-diagram for a finitely generated group relative to a word hyperbolic group. In her thesis [1], Emina Alibegović did the same relative to a limit group.

Corollary 1.23. *Abelian subgroups of limit groups are fg and free.*

We first need a lemma.

Lemma 1.24. *Suppose that Γ is a limit group with factor set $\{q_i : \Gamma \to G_i\}$. If H is a (not necessarily fg) subgroup of Γ such that every homomorphism $H \to \mathbb{F}$ factors through some $q_i|H$ (pre-compositions by automorphisms of Γ not needed) then, for some i, $q_i|H$ is injective.*

Proof. Suppose not and let $1 \neq h_i \in Ker(q_i|H)$. Since Γ is a limit group, there is $f \in Hom(\Gamma, \mathbb{F})$ that is injective on $\{1, h_1, \cdots, h_n\}$. On the other hand, $f|H$ factors through some $q_i|H$ and so $h_i = 1$, a contradiction. □

Proof of Corollary 1.23. Let A be an abelian subgroup of a limit group Γ. Since residually free groups are torsion free, it is enough to show that A is fg. It follows from the definition that, for $\alpha \in Mod(\Gamma)$, there is a finitely generated subgroup A_α of A and a retraction $r_\alpha : \Gamma \to A_\alpha$ such that $\alpha|A$ is trivial[11] on $A \cap Ker(r_\alpha)$. See Remark 1.12.

Consider the homomorphism $\Pi_{\alpha \in Mod(\Gamma)} r_\alpha : \Gamma \to \Pi_\alpha A_\alpha$. Since Γ is fg, the image of Πr_α is fg. Hence, $A = A_0 \oplus A_1$ where A_1 is fg and each r_α is trivial on A_0. Let $\{q_i : \Gamma \to \Gamma_i\}$ be a factor set for Γ with each Γ_i a limit group. By Lemma 1.24, A_0 injects into some Γ_i. Since $Hom(\Gamma_i, \mathbb{F}) \subsetneq Hom(\Gamma, \mathbb{F})$, we may conclude by induction that A_0 and hence A is fg. □

In Section 6, we will also show:

Theorem 1.25. *For an fg group G, the following are equivalent.*

1. *G is a CLG.*

[11] agrees with the restriction of an inner automorphism of Γ.

2. G is ω-residually free.

3. G is a limit group.

The fact that ω-residually free groups are CLG's is due to O. Kharlampovich and A. Myasnikov [11]. V.N. Remeslennikov [21] showed that limit groups act freely on $\mathbb{R}^n$-trees, also see [8]. Kharlampovich-Myasnikov [10] prove that limit groups act freely on $\mathbb{Z}^n$-trees where $\mathbb{Z}^n$ is lexicographically ordered. Remeslennikov [20] also demonstrated that 2-residually free groups are ω-residually free.

2 The Main Proposition

Definition 2.1. An fg group is *generic* if it is torsion free, freely indecomposable, non-abelian, and not a closed surface group.

The Main Theorem will follow from the next proposition.

Main Proposition. *Generic limit groups have factor sets.*

Before proving this proposition, we show how it implies the Main Theorem.

Definition 2.2. Let G and G' be fg groups. The minimal number of generators for G is denoted $\mu(G)$. We say that G is *simpler* than G' if there is an epimorphism $G' \to G$ and either $\mu(G) < \mu(G')$ or $\mu(G) = \mu(G')$ and $Hom(G, \mathbb{F}) \subsetneq Hom(G', \mathbb{F})$.

Remark 2.3. It follows from Lemma 1.8 that every sequence $\{G_i\}$ with G_{i+1} simpler than G_i is finite.

Definition 2.4. If G is an fg group, then by $RF(G)$ denote the universal residually free quotient of G, i.e. the quotient of G by the (normal) subgroup consisting of elements killed by every homomorphism $G \to \mathbb{F}$.

Remark 2.5. $Hom(G, \mathbb{F}) = Hom(RF(G), \mathbb{F})$ and for every proper quotient G' of $RF(G)$, $Hom(G', \mathbb{F}) \subsetneq Hom(G, \mathbb{F})$.

The Main Proposition implies the Main Theorem. Suppose that G is an fg group that is not free. By Remark 2.3, we may assume that the Main Theorem holds for groups that are simpler than G. By Remark 2.5, we may assume that G is residually free, and so also torsion free. Examples 1.3

and 1.4 show that the Main Theorem is true for abelian and closed surface groups. If $G = U * V$ with U non-free and freely indecomposable and with V non-trivial, then U is simpler than G. So, G has a factor set $\{q_i : U \to L_i\}$, and $\{q_i * Id_V : U * V \to L_i * V\}$ is a factor set for G.

If G is not a limit group, then there is a finite subset $\{g_i\}$ of G such that any homomorphism $G \to \mathbb{F}$ kills one of the g_i. We then have a factor set $\{G \to H_i := G/\langle\langle g_i \rangle\rangle\}$. Since $Hom(H_i, \mathbb{F}) \subsetneq Hom(G, \mathbb{F})$, by induction the Main Theorem holds for H_i and so for G.

If G is generic and a limit group, then the Main Proposition gives a factor set $\{q_i : G \to G_i\}$ for G. Since G is residually free, each G_i is simpler than G. We are assuming that the Main Theorem then holds for each G_i and this implies the result for G. □

3 Review: Measured laminations and $\mathbb{R}$-trees

The proof of the Main Proposition will use a theorem of Sela describing the structure of certain real trees. This in turn depends on the structure of measured laminations. In Section 7, we will give an alternate approach that only uses the lamination results. First these concepts are reviewed. A more leisurely review with references is [2].

3.1 Laminations

Definition 3.1. A *measured lamination* Λ on a simplicial 2-complex K consists of a closed subset $|\Lambda| \subset |K|$ and a *transverse measure* μ. $|\Lambda|$ is disjoint from the vertex set, intersects each edge in a Cantor set or empty set, and intersects each 2-simplex in 0, 1, 2, or 3 families of straight line segments spanning distinct sides. μ assigns a non-negative number $\int_I \mu$ to every interval I in an edge whose endpoints are outside $|\Lambda|$. There are two conditions:

1. **(compatibility)** If two intervals I, J in two sides of the same triangle Δ intersect the same components of $|\Lambda| \cap \Delta$ then $\int_I \mu = \int_J \mu$.

2. **(regularity)** μ restricted to an edge is equivalent under a "Cantor function" to the Lebesgue measure on an interval in $\mathbb{R}$.

A path component of $|\Lambda|$ is a *leaf*.

Two measured laminations on K are considered equivalent if they assign the same value to each edge.

Proposition 3.2 (Morgan-Shalen [16]). *Let Λ be a measured lamination on K. Then*

$$\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_k$$

so that each Λ_i is either minimal (each leaf is dense in $|\Lambda_i|$) or simplicial (each leaf is compact, a regular neighborhood of $|\Lambda_i|$ is an I-bundle over a leaf and $|\Lambda_i|$ is a Cantor set subbundle).

There is a theory, called the *Rips machine*, for analyzing minimal measured laminations. It turns out that there are only 3 qualities.

Example 3.3 *(Surface type).* Let S be a compact hyperbolic surface (possibly with totally geodesic boundary). If S admits a pseudoAnosov homeomorphism then it also admits *filling measured geodesic laminations* – these are measured laminations Λ (with respect to an appropriate triangulation) such that each leaf is a biinfinite geodesic and all complementary components are ideal polygons or crowns. Now to get the model for a general surface type lamination attach finitely many annuli $S^1 \times I$ with lamination $S^1 \times$ (Cantor set) to the surface along arcs transverse to the geodesic lamination. If these additional annuli do not appear then the lamination is of *pure surface type.* See Figure 1.

Example 3.4 *(Toral type).* Fix a closed interval $I \subset \mathbb{R}$, a finite collection of pairs (J_i, J_i') of closed intervals in I, and isometries $\gamma_i : J_i \to J_i'$ so that:

1. If γ_i is orientation reversing then $J_i = J_i'$ and the midpoint is fixed by γ_i.
2. The length of the intersection of all J_i and J_i' (over all i) is more than twice the translation length of each orientation preserving γ_i and the fixed points of all orientation reversing γ_i are in the middle third of the intersection.

Now glue a foliated band for each pair (J_i, J_i') so that following the band maps J_i to J_i' via γ_i. Finally, using Cantor functions blow up the foliation to a lamination. There is no need to explicitly allow adding annuli as in the surface case since they correspond to $\gamma_i = Id$. The subgroup of $Isom(\mathbb{R})$ generated by the extensions of the γ_i's is the *Bass group.* The lamination is minimal iff its Bass group is not discrete.

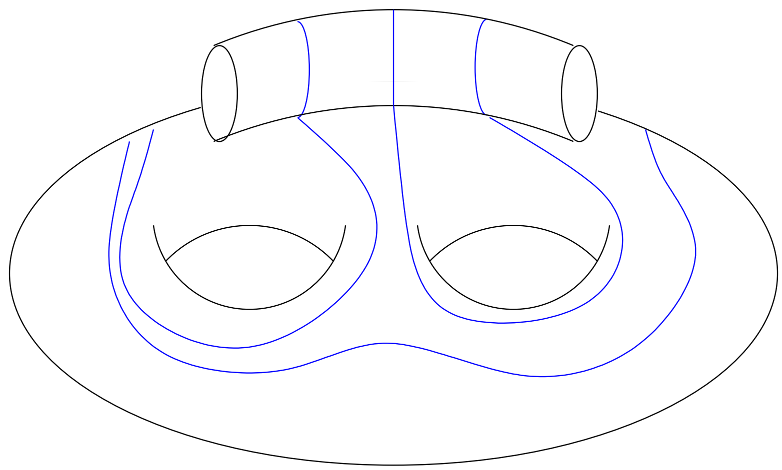

Figure 1: A surface with an additional annulus and some pieces of leaves.

Example 3.5 *(Thin type).* This is the most mysterious type of all. It was discovered by Gilbert Levitt, see [12]. In the "pure" case (with no annuli attached) the leaves are 1-ended trees (so this type naturally lives on a 2-complex, not on a manifold). By performing certain moves (sliding, collapsing) that don't change the homotopy type (respecting the lamination) of the complex one can transform it to one that contains a (thin) band. This band induces a non-trivial free product decomposition of $\pi_1(K)$, assuming that the component is a part of a resolution of a tree (what's needed is that loops that follow leaves until they come close to the starting point and then they close up are non-trivial in π_1).

In the general case we allow additional annuli to be glued, just like in the surface case. Leaves are then 1-ended trees with circles attached. Again, if there are no additional annuli then the lamination is *pure*.

Theorem 3.6 ("Rips machine"). *Let Λ be a measured lamination on a finite 2-complex K, and let Λ_i be a minimal component of Λ. There is a neighborhood N (we refer to it as a "standard" neighborhood) of $|\Lambda_i|$, a finite 2-complex N' with measured lamination Λ' as in one of 3 model examples, and there is a π_1-isomorphism $f : N \to N'$ such that $f^*(\Lambda') = \Lambda$.*

We refer to Λ_i as being of *surface*, *toral*, or *thin* type.

3.2 Dual trees

Let G be an fg group and let $\hat{K}$ be a simply connected 2-dimensional simplicial G-complex so that, for each simplex Δ of $\hat{K}$, $Stab(\Delta) = Fix(\Delta)$.[12] Let $\hat{\Lambda}$ be a G-invariant measured lamination in $\hat{K}$. There is an associated real G-tree $T(\hat{\Lambda})$ constructed as follows. Consider the pseudo-metric on $\hat{K}$ obtained by minimizing the $\hat{\Lambda}$-length of paths between points. The real tree $T(\hat{\Lambda})$ is the associated metric space[13]. There is a natural map $\hat{K} \to T(\hat{\Lambda})$ and we say that $(\hat{K}, \hat{\Lambda})$ is a *model* for $T(\hat{\Lambda})$ if

- for each edge $\hat{e}$ of $\hat{K}$, $T(\hat{\Lambda} \mid \hat{e}) \to T(\hat{\Lambda})$ is an isometry (onto its image) and
- the quotient $\hat{K}/G$ is compact.

If a tree T admits a model $(\hat{K}, \hat{\Lambda})$, then we say that T is *geometric* and that T is *dual* to $(\hat{K}, \hat{\Lambda})$. This is denoted $T = Dual(\hat{K}, \hat{\Lambda})$. We will use the quotient $(K, \Lambda) := (\hat{K}, \hat{\Lambda})/G$ with simplices decorated (or labeled) with stabilizers to present a model and sometimes abuse notation by calling (K, Λ) a model for T.

Remark 3.7. Often the G-action on $\hat{K}$ is required to be free. We have relaxed this condition in order to be able to consider actions of fg groups. For example, if T is a minimal[14], simplicial G-tree (with the metric where edges have length one[15]) then there is a lamination $\hat{\Lambda}$ in T such that $Dual(T, \hat{\Lambda}) = T$. [16]

If S and T are real G-trees, then an equivariant map $f : S \to T$ is a *morphism* if every compact segment of S has a finite partition such that the restriction of f to each element is an isometry or trivial[17].

If S is a real G-tree with G fp, then there is a geometric real G-tree T and a morphism $f : T \to S$. The map f is obtained by constructing an equivariant map to S from the universal cover of a 2-complex with fundamental group

[12] $Stab(\Delta) := \{g \in G \mid g\Delta = \Delta\}$ and $Fix(\Delta) := \{g \in G \mid gx = x, x \in \Delta\}$

[13] identify points of pseudo-distance 0

[14] no proper invariant subtrees

[15] This is called the *simplicial metric* on T.

[16] The metric and simplicial topologies on T don't agree unless T is locally finite. But, the action of G is by isomorphisms in each structure. So, we will be sloppy and ignore this distinction.

[17] has image a point

G. In general, if $(\hat{K}, \hat{\Lambda})$ is a model for T and if $T \to S$ is a morphism then the composition $\hat{K} \to T \to S$ is a *resolution* of S.

3.3 The structure theorem

Here we discuss a structure theorem (see Theorem 3.13) of Sela for certain actions of an fg torsion free group G on real trees. Among other restrictions, the actions we consider will be required to be stable[18], have abelian (non-degenerate) arc stabilizers, and have trivial tripod[19] stabilizers. There is a short list of basic examples.

Example 3.8 *(Pure surface type).* A real G-tree T is *of pure surface type* if it is dual to the universal cover of (K, Λ) where K is a compact surface and Λ is of pure surface type. We will usually use the alternate model where boundary components are crushed to points and are labeled $\mathbb{Z}$.

Example 3.9 *(Linear).* The tree T is *linear* if G is abelian, T is a line and there an epimorphism $G \to \mathbb{Z}^n$ such that G acts on T via a free $\mathbb{Z}^n$-action on T. In particular, T is geometric and is dual to $(\hat{K}, \hat{\Lambda})$ where $\hat{K}$ is the universal cover of the 2-skeleton of an n-torus K. For simplicity, we often complete K with its lamination to the whole torus. This is a special case of a toral lamination.

Example 3.10 *(Pure thin).* The tree T is *pure thin* if it is dual to the universal cover of a finite 2-complex K with a pure thin lamination Λ. If T is pure thin then $G \cong F * V_1 * \cdots * V_m$ where F is non-trivial and fg free and $\{V_1, \cdots, V_m\}$ represents the conjugacy classes of point stabilizers in T.

Example 3.11 *(Simplicial).* The tree T is *simplicial* if it is dual to $(\hat{K}, \hat{\Lambda})$ where all leaves of $\Lambda := \hat{\Lambda}/G$ are compact. If T is simplicial it is convenient to crush the leaves and complementary components to points in which case $\hat{K}$ becomes a tree isomorphic to T.

If $\mathcal{K}$ is a graph of 2-complexes with underlying graph of groups $\mathcal{G}$[20] then there is a simplicial $\pi_1(\mathcal{G})$-space $\hat{K}(\mathcal{K})$ obtained by gluing copies of $\hat{K}_e \times I$ and $\hat{K}_v$'s equipped with a simplicial $\pi_1(\mathcal{G})$-map $\hat{K}(\mathcal{K}) \to T(\mathcal{G})$ that crushes to points copies of $\hat{K}_e \times \{point\}$ as well as the $\hat{K}_v$'s.

[18] every (non-degenerate) arc in T contains a subarc α with the property that every subarc of α has the same stabilizer as α

[19] a cone on 3 points

[20] for each bonding map $\phi_e : G_e \to G_v$ there are simplicial G_e- and G_v-complexes $\hat{K}_e$ and $\hat{K}_v$ together with a ϕ_e-equivariant simplicial map $\Phi_e : \hat{K}_e \to \hat{K}_v$

Definition 3.12. A real G-tree is *very small* if it is non-trivial[21], minimal, stable, has abelian (non-degenerate) arc stabilizers, and has trivial (non-degenerate) tripod stabilizers.

Theorem 3.13 ([29, Section 3]). *Let T be a real G-tree. Suppose that G is generic and that T is very small. Then, T is geometric.*

Moreover, there is a model for T that is a graph of spaces such that each edge space is a point with non-trivial abelian stabilizer and each vertex space with restricted lamination is either

- (point) *a point with non-trivial stabilizer,*
- (linear) *a non-faithful action of an abelian group on the (2-skeleton of the) universal cover of a torus with an irrational* [22] *lamination, or*
- (surface) *a faithful action of a free group on the universal cover of a surface with non-empty boundary (represented by points with $\mathbb{Z}$-stabilizer) with a lamination of pure surface type.*

Remark 3.14. For an edge space $\{point\}$, the restriction of the lamination to $\{point\} \times I$ may or may not be empty. It can be checked that between any two points in models as in Theorem 3.13 there are Λ-length minimizing paths. Thin pieces do not arise because we are assuming our group is freely indecomposable.

Remark 3.15. Theorem 3.13 holds more generally if the assumption that G is freely indecomposable is replaced by the assumption that G is freely indecomposable rel point stabilizers, i.e. if $\mathcal{V}$ is the subset of G of elements acting elliptically[23] on T, then G cannot be expressed non-trivially as $A * B$ with all $g \in \mathcal{V}$ conjugate into $A \cup B$.

We can summarize Theorem 3.13 by saying that T is a non-trivial finite graph of simplicial trees, linear trees, and trees of pure surface type (over trivial trees). See Figure 2.

Corollary 3.16. *If G and T satisfy the hypotheses of Theorem 3.13, then G admits a non-trivial* GAD *Δ. Specifically, Δ may be taken to be the* GAD *induced by the boundary components of the surface vertex spaces and the*

[21] no fixed point
[22] no essential loops in leaves
[23] fixing a point

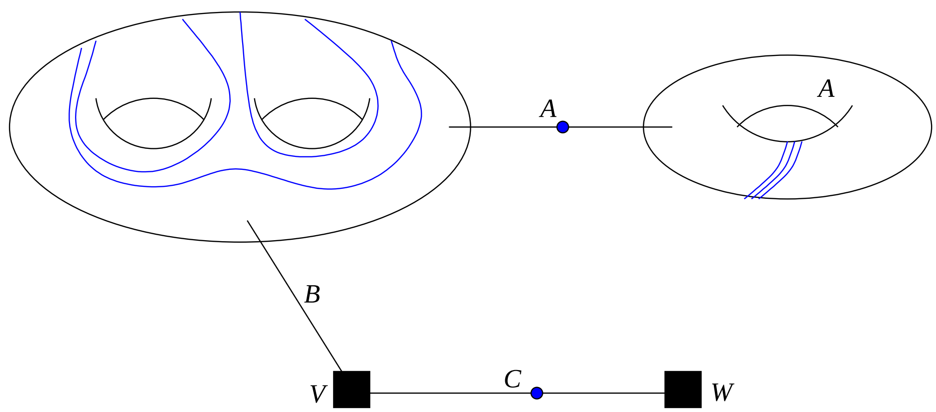

Figure 2: A model with a surface vertex space, a linear vertex space, and 2 rigid vertex spaces (the black boxes). The groups A, B and C are abelian with A and B infinite cyclic. Pieces of some leaves are also indicated by wavy lines and dots. For example, the dot on the edge labeled C is one leaf in a Cantor set of leaves.

simplicial edges of the model. The surface vertex spaces give rise to the QH-*vertices of* Δ *and the linear vertex spaces give rise to the abelian vertices of* Δ.

3.4 Spaces of trees

Let G be a non-trivial fg group and let $\mathcal{A}(G)$ be the set of minimal real G-trees endowed with the Gromov topology. Recall, see [17, 18, 4], that in the Gromov topology $\lim\{(T_n, d_n)\} = (T, d)$ if and only if: for any finite subset K of T, any $\epsilon > 0$, and any finite subset P of G, for sufficiently large n, there are subsets K_n of T_n and bijections $f_n : K_n \to K$ such that

$$|d(gf_n(s_n), f_n(t_n)) - d_n(gs_n, t_n)| < \epsilon$$

for all $s_n, t_n \in K_n$ and all $g \in P$. Intuitively, larger and larger pieces of the limit tree with their restricted actions appear in nearby trees.

Let $\mathcal{PA}(G)$ be the set of non-trivial real G-trees modulo homothety, i.e. $(T, d) \sim (T, \lambda d)$ for $\lambda > 0$. Fix a basis for $\mathbb{F}$ and let $T_{\mathbb{F}}$ be the corresponding Cayley graph. Give $T_{\mathbb{F}}$ the simplicial metric. So, a non-trivial homomorphism $f : G \to \mathbb{F}$ determines $T_f \in \mathcal{PA}(G)$. The space of interest is the closure $\mathcal{T}(G)$ of $\{T_f \mid 1 \neq f \in Hom(G, \mathbb{F})\}$ in $\mathcal{PA}(G)$.

Proposition 3.17 ([30]). *Every sequence of non-trivial homomorphisms from G to $\mathbb{F}$ has a subsequence $\{f_n\}$ such that $\lim T_{f_n} = T$ in $\mathcal{T}(G)$. Further,*

1. *T is non-trivial.*

2. *If T is not a line, then $\underrightarrow{Ker}\ f_n$ is precisely the kernel $Ker(T)$ of the action of G on T.*

3. *The stabilizer $Stab_{G/Ker(T)}(I)$ in $G/Ker(T)$ of every (non-degenerate and perhaps non-compact) arc $I \subset T$ is free abelian. Moreover, if $Fix_{G/Ker(T)}(I) \neq 1$ then for every subarc $J \subset I$ we have equality $Fix_{G/Ker(T)}(I) = Fix_{G/Ker(T)}(J)$. In particular, T is stable.*

4. *The stabilizer in $G/Ker(T)$ of every tripod is trivial.*

5. *T is a line iff almost all f_n have non-trivial abelian image.*

Proof. The first statement is easy if the sequence contains infinitely many homomorphisms with abelian image. Otherwise it follows from Paulin's Convergence Theorem [17]. The further items are exercises in Gromov convergence. □

Caution. Sela goes on the claim that stabilizers of minimal components of the limit tree are trivial (see Lemma 1.6 of [30]). However, it is possible to construct limit actions on the amalgam of a rank 2 free group F_2 and $\mathbb{Z}^3$ over $\mathbb{Z}$ where one of the generators of $\mathbb{Z}^3$ is glued to the commutator c of basis elements of F_2 and where the $\mathbb{Z}^3$ acts non-simplicially on a linear subtree with c acting trivially on the subtree but not in the kernel of the action. As a result, some of his arguments, though easily completed, are not fully complete.

Corollary 3.18.

1. *$\mathcal{T}(G)$ is compact.*

2. *The subspace $\mathcal{L}(G)$ of $\mathcal{T}(G)$ consisting of linear trees is clopen[24].*

3. *For $g \in G$, $U(g) := \{T \in \mathcal{T}(G) \setminus \mathcal{L}(G) \mid g \in Ker(T)\}$ is clopen.*

Remark 3.19. There is another common topology on $\mathcal{A}(G)$, the length topology. For $T \in \mathcal{A}(G)$ and $g \in G$, let $\|g\|_T$ denote the minimum distance that g translates a point of T. The length topology is induced by the map

[24]both open and closed

$\mathcal{A}(G) \to [0, \infty)^G$, $T \mapsto (\|g\|_T)_{g \in G}$. The subspace $\mathcal{L}(G)$ is clopen in $\mathcal{PA}(G)$ with respect to either the Gromov topology or length topology. It follows from work of Paulin [18] that the closures of $\{T_f\}$ in $\mathcal{PA}(G) \setminus \mathcal{L}(G)$ with respect to the two topologies also agree and are homeomorphic (by the identity function). Finally, with respect to either topology, $\mathcal{L}(G)$ is homeomorphic to

$$[Hom(Ab(G), \mathbb{R}) \setminus \{0\}]/(0, \infty)$$

with its natural topology. In particular, the the closures of $\{T_f\}$ in $\mathcal{PA}(G)$ with respect to the two topologies agree and are homeomorphic.

Remark 3.20. $\mathcal{L}(G)$ is a real projective space and, for $g \in G$, the set $\{T \in \mathcal{L}(G) \mid g \in Ker(T)\}$ is a subprojective space, so is closed but not generally open.

4 Proof of the Main Proposition

To warm up, we first prove the Main Proposition under the additional assumption that Γ has only trivial abelian splittings, i.e. every simplicial Γ-tree with abelian edge stabilizers has a fixed point. This proof is then modified to apply to the general case.

Proposition 4.1. *Suppose that Γ is a generic limit group and has only trivial abelian splittings*[25]. *Then, Γ has a factor set.*

Proof. Let $T \in \mathcal{T}(\Gamma)$. By Proposition 3.17, either $\Gamma/Ker(T)$ is non-generic or satisfies the hypotheses of Theorem 3.13. In any case, by Corollary 3.16, $\Gamma/Ker(T)$ admits a non-trivial abelian splitting. In particular, $Ker(T)$ is non-trivial. Choose non-trivial $k_T \in Ker(T)$. By Corollary 3.18, $\{\mathcal{L}(\Gamma)\} \cup \{U(k_T) \mid T \in \mathcal{T}(\Gamma) \setminus \mathcal{L}(\Gamma)\}$ is an open cover of $\mathcal{T}(\Gamma)$. Let $\{\mathcal{L}(\Gamma)\} \cup \{U(k_i)\}$ be a finite subcover. By definition, $\{\Gamma \to Ab(\Gamma)\} \cup \{q_i : \Gamma \to \Gamma/\langle\langle k_i \rangle\rangle\}$ is a factor set. ∎

The key to the proof of the general case is Sela's notion of a *short* homomorphism, a concept which we now define.

[25]By Proposition 3.17 and Corollary 3.16, limit groups have non-trivial abelian splittings. The purpose of this proposition is to illustrate the method in this simpler (vacuous) setting.

Definition 4.2. Let G be an fg group. Two elements f and f' in $Hom(G, \mathbb{F})$ are *equivalent*, denoted $f \sim f'$, if there is $\alpha \in Mod(G)$ and an element $c \in \mathbb{F}$ such that $f' = i_c \circ f \circ \alpha$.[26] Fix a set B of generators for G and by $|f|$ denote $\max_{g \in B} |f(a)|$ where, for elements of $\mathbb{F}$, $|\cdot|$ indicates word length. We say that f is *short* if, for all $f' \sim f$, $|f| \leq |f'|$.

Here is another exercise in Gromov convergence. See [30, Claim 5.3] and [2, Theorem 7.4].

Exercise 11. *Suppose that G is generic, $\{f_i\}$ is a sequence in $Hom(G, \mathbb{F})$, and $\lim T_{f_i} = T$ in $\mathcal{T}(G)$. Then, either*

- *$Ker(T)$ is non-trivial or,*
- *eventually f_i is not short.*

The idea is that if the first bullet does not hold, then the GAD of G given by Corollary 3.16 can be used to find elements of $Mod(G)$ that shorten f_i for i large. Let $\mathcal{T}'(G)$ be the closure in $\mathcal{T}(G)$ of $\{T_f \mid f \text{ is short in } Hom(G, \mathbb{F})\}$. By Corollary 3.18(1), $\mathcal{T}'(G)$ is compact.

Proof of the Main Proposition. Let $T \in \mathcal{T}'(\Gamma)$. By Exercise 11, $Ker(T)$ is non-trivial. Choose non-trivial $k_T \in Ker(T)$. By Corollary 3.18, $\{\mathcal{L}(\Gamma)\} \cup \{U(k_T) \mid T \in \mathcal{T}'(\Gamma) \setminus \mathcal{L}(\Gamma)\}$ is an open cover of $\mathcal{T}'(\Gamma)$. Let $\{\mathcal{L}(\Gamma)\} \cup \{U(k_i)\}$ be a finite subcover. By definition, $\{\Gamma \to Ab(\Gamma)\} \cup \{q_i : \Gamma \to \Gamma/\langle\langle k_i \rangle\rangle\}$ is a factor set. □

JSJ-decompositions will be used to prove Theorem 1.25, so we digress.

5 Review: JSJ-theory

Some familiarity with JSJ-theory is assumed. The reader is referred to Rips-Sela [22], Dunwoody-Sageev [5], Fujiwara-Papasoglou [6]. For any generic group G consider the class of 1-edge splittings such that:

(JSJ 1) the edge group is abelian,

(JSJ 2) the edge group is primitive[27], and

[26] i_c is conjugation by c

[27] closed under taking roots

(JSJ 3) every non-cyclic abelian subgroup $A \subset G$ is elliptic.

We observe that

- Any two such splittings are hyperbolic-hyperbolic[28] or elliptic-elliptic[29](a hyperbolic-elliptic pair implies that one splitting can be used to refine the other. Since the hyperbolic edge group is necessarily cyclic by (JSJ 3), this refinement gives a free product decomposition of G).
- A hyperbolic-hyperbolic pair has both edge groups cyclic and yields a GAD of G with a QH-vertex group.
- An elliptic-elliptic pair has a common refinement that satisfies (JSJ 1)–(JSJ 3) and whose set of elliptics is the intersection of the sets of elliptics in the given splittings.

Given a GAD Δ of G, we say that $g \in G$ is Δ-*elliptic* if it is conjugate to a an element v of a vertex group V of Δ and further,

- If V is QH then v is a multiple of a boundary component.
- If V is abelian then $v \in \overline{P}(V)$.

The idea is that Δ gives rise to a family of 1-edge splittings coming from edges of the decomposition, from simple closed curves in QH-vertex groups, and from subgroups A' of an abelian vertex A that contain $P(A)$ and with $A/A' \cong \mathbb{Z}$. An element is Δ-elliptic iff it is elliptic with respect to all these 1-edge splittings. Conversely, any finite collection of 1-edge splittings satisfying (JSJ 1)–(JSJ 3) gives rise to a GAD whose set of elliptics is precisely the intersection of the set of elliptics in the collection.

Definition 5.1. An *abelian JSJ-decomposition of* G is a GAD whose elliptic set is the intersection of elliptics in the family of *all* 1-edge splittings satisfying (JSJ 1)–(JSJ 3).

For example, $G = F \times \mathbb{Z}$ does not have any splittings satisfying (JSJ 1)-(JSJ 3) so the abelian JSJ is G itself. Of course, G does have (many) abelian splittings (but they don't satisfy (JSJ 3)).

[28]each edge group of each tree contains an element not fixing a point of the other tree
[29]each edge group of each tree fixes a point of the other tree

Remark 5.2. Since abelian subgroups of limit groups are fg (Corollary 1.23), (JSJ 3) implies that for a limit group all the splittings that we will consider have the property that non-cyclic abelian subgroups are conjugate into a vertex group.

To show that a group G admits an abelian JSJ-decomposition it is necessary to show that there is a bound to the complexity of the GAD's arising from finite collections of 1-edge splittings satisfying (JSJ 1)–(JSJ 3). If G were fp the results of [3] would suffice. Since we don't know yet that limit groups are fp, another technique is needed. Following Sela, we use acylindrical accessibility.

Definition 5.3. A simplicial G-tree T is *n-acylindrical* if, for non-trivial $g \in G$, the diameter in the simplicial metric of the sets $Fix(g)$ is bounded by n. It is *acylindrical* if it is n-acylindrical for some n.

Theorem 5.4 (Acylindrical Accessibility: Sela [29], Weidmann [34]). *Let G be a non-cyclic freely indecomposable fg group and let T be a minimal k-acylindrical simplicial G-tree. Then, T/G has at most $1 + 2k(rank\ G - 1)$ vertices.*

The explicit bound in Theorem 5.4 is due to Richard Weidmann. For limit groups, 1-edge splittings satisfying (JSJ 1)–(JSJ 3) are 2-acylindrical and finitely many such splittings give rise to GAD's that can be arranged to be 2-acylindrical. Theorem 5.4 can then be applied to show that abelian JSJ-decomposition exist.

Theorem 5.5 ([30]). *Limit groups admit abelian JSJ-decompositions.*

Definition 5.6. Let Δ be a 1-edge splitting of a group G with abelian edge group C. Let z be an element of the centralizer $Z_G(C)$ of C in G. The automorphism α_z of G, called the *Dehn twist in z*, is determined as follows. There are two cases.

1. $\Delta = A *_C B$:

$$\alpha_z(g) = \begin{cases} g, & \text{if } g \in A; \\ zgz^{-1}, & \text{if } g \in B. \end{cases}$$

2. $\Delta = A*_C$

$$\alpha_z(g) = \begin{cases} g, & \text{if } g \in A; \\ gz, & \text{if } g \text{ is the stable letter.} \end{cases}$$

If Δ is as in Cases (1) and (2) and if A is abelian then a *generalized Dehn twist* is a Dehn twist or an automorphism of G that restricts to a unimodular automorphism of A and that is the identity when restricted to the edge groups incident to A and when restricted to B if B exists. $Mod(\Delta)$ is the subgroup of $Mod(G)$ generated by these generalized Dehn twists.

Exercise 12.

- *$Mod(G)$ is generated by inner automorphisms together with generalized Dehn twists.*
- *If G is a limit group, then $Mod(G)$ is generated by inner automorphisms together with generalized Dehn twists associated to 1-edge splittings of G satisfying (JSJ 1)–(JSJ 3). See [30, Lemma 2.1]. In fact, the only generalized Dehn twists that are not Dehn twists can be taken to be with respect to a splitting of the form $A *_C B$ where $A = C \oplus \mathbb{Z}$.*

Remark 5.7. If B is a rigid vertex group of an abelian JSJ-decomposition of a group G and if $\alpha \in Mod(G)$, then there is an element of g such that $\alpha|B = i_g|B$. Indeed, B is elliptic in any 1-edge splitting of G and so the statement is true for generators of $Mod(G)$.

6 Limit groups are CLG's

In this section, we show that limit groups are CLG's and complete the proof of Theorem 1.25.

Lemma 6.1. *Limit groups are CLG's*

Proof. Let Γ be a limit group, which we may assume is generic. Let $\{f_i\}$ be a sequence in $Hom(\Gamma, \mathbb{F})$ such that f_i is injective on elements of length at most i (with respect to some finite generating set for Γ). Define $\hat{f}_i$ to be a short map equivalent to f_i. According to Exercise 11, $q : \Gamma \to \Gamma' := \Gamma / \underrightarrow{Ker}\ \hat{f}_i$ is a proper epimorphism, and so by induction we may assume that Γ' is a CLG.

Let Δ be an abelian JSJ-decomposition of Γ. We will show that q and Δ satisfy the conditions in Definition 1.14. The key observations are these.

- If A is a peripheral subgroup of an abelian vertex of Δ, then A is elliptic in all 1-edge splittings of Γ. In particular, Dehn twists, hence all elements of $Mod(\Gamma)$, when restricted to A are trivial[30]. Also, by Corollary 1.23, abelian vertex groups of Δ are CLG's.

- Elements of $Mod(\Gamma)$ when restricted to edge groups of Δ are trivial. Since Γ is a limit group, each edge group is a maximal abelian subgroup in at least one of the two adjacent vertex groups. See Exercise 6.

- The q-image of a QH-vertex group Q of Δ is non-abelian. Indeed, suppose that Q is a QH-vertex group of Δ and that $q(Q)$ is abelian. Then, eventually $\hat{f}_i(Q)$ is abelian. QH-vertex groups of abelian JSJ-decompositions are canonical, and so every element of $Mod(\Gamma)$ preserves Q up to conjugacy. Hence, eventually $f_i(Q)$ is abelian, contradicting the triviality of $\underrightarrow{Ker}\ f_i$.

- Elements of $Mod(\Gamma)$ when restricted to envelopes of rigid vertex groups of Δ are trivial. Since $\underrightarrow{Ker}\ f_i$ is trivial, q is injective on these envelopes. In particular, rigid vertex groups of Δ are CLG's.

□

Proof of Theorem 1.25. (1) $\implies$ (2) $\implies$ (3) were exercises. (3) $\implies$ (1) is the content of Lemma 6.1. □

7 A more geometric approach

In this section, we show how to derive the Main Proposition using Rips theory for fp groups in place of the structure theory of actions of fg groups on real trees.

Definition 7.1. Let K be a finite 2-complex with a measured lamination (Λ, μ). The *length of* Λ, denoted $\|\Lambda\|$, is the sum $\Sigma_e \int_e \mu$ over the edges e of K.

If $\phi : \tilde{K} \to T$ is a resolution, then $\|\phi\|_K$ is the length of the induced lamination Λ_ϕ. Suppose that K is a 2-complex for G.[31] Recall that $T_{\mathbb{F}}$ is a

[30] Recall our convention that "trivial" means "agrees with the restriction of an inner automorphism". Alternatively, we could view elements of $Mod(\Gamma)$ as inducing the trivial outer automorphism on A.

[31] i.e. the fundamental group of K is identified with the group G

Cayley graph for $\mathbb{F}$ with respect to a fixed basis and that from a homomorphism $f : G \to \mathbb{F}$ a resolution $\phi : (\tilde{K}, \tilde{K}^{(0)}) \to (T_{\mathbb{F}}, T_{\mathbb{F}}^{(0)})$ can be constructed, see [3]. The resolution ϕ depends on a choice of images of a set of orbit representatives of vertices in $\tilde{K}$. We will always choose ϕ to minimize $\|\phi\|_K$ over this set of choices. With this convention, we define $\|f\|_K := \|\phi\|_K$.

Lemma 7.2. *Let K_1 and K_2 be finite 2-complexes for G. There is a number $B = B(K_1, K_2)$ such that, for all $f \in Hom(G, \mathbb{F})$,*

$$B^{-1} \cdot \|f\|_{K_1} \leq \|f\|_{K_2} \leq B \cdot \|f\|_{K_1}.$$

Proof. Let $\phi_1 : \tilde{K}_1 \to T_{\mathbb{F}}$ be a resolution such that $\|\phi_1\|_{K_1} = \|f\|_{K_1}$. Choose an equivariant map $\psi^{(0)} : \tilde{K}_2^{(0)} \to \tilde{K}_1^{(0)}$ between 0-skeleta. Then, $\phi_1\psi^{(0)}$ determines a resolution $\phi_2 : \tilde{K}_2 \to T_{\mathbb{F}}$. Extend $\psi^{(0)}$ to a cellular map $\psi^{(1)} : \tilde{K}_2^{(1)} \to \tilde{K}_1^{(1)}$ between 1-skeleta. Let B_2 be the maximum over the edges e of the simplicial length of the path $\psi^{(1)}(e)$ and let E_2 be the number of edges in K_2. Then,

$$\|f\|_{K_2} \leq \|\phi_2\|_{K_2} \leq B_2 N_2 \|\phi_1\|_{K_1} = B_2 N_2 \|f\|_{K_1}.$$

The other inequality is similar. ☐

Recall that in Definition 4.2, we defined another length $|\cdot|$ for elements of $Hom(G, \mathbb{F})$.

Corollary 7.3. *Let K be a finite 2-complex for G. Then, there is a number $B = B(K)$ such that for all $f \in Hom(G, \mathbb{F})$*

$$B^{-1} \cdot |f| \leq \|f\|_K \leq B \cdot |f|.$$

Proof. If B is the fixed finite generating set for G and if R_B is the wedge of circles with fundamental group identified with the free group on B, then complete R_B to a 2-complex for G by adding finitely many 2-cells and apply Lemma 7.2. ☐

Remark 7.4. Lemma 7.2 and its corollary allow us to be somewhat cavalier with our choices of generating sets and 2-complexes.

Exercise 13. *The space of (nonempty) measured laminations on K can be identified with the closed cone without 0 in $\mathbb{R}_+^E$, where E is the set of edges of K, given by the triangle inequalities for each triangle of K. The projectivized space $\mathcal{PML}(K)$ is compact.*

Exercise 14. *If* $\lim_{i\to\infty} T_{f_i} = T$ *and* $\lim_{i\to\infty} \Lambda_{f_i} = \Lambda$ *then there is a resolution that sends lifts of leaves of* Λ *to points of* T *and is monotonic (Cantor function) on edges of* $\tilde{K}$*.*

Definition 7.5. An element f of $Hom(G, \mathbb{F})$ is K*-short* if $\|f\|_K \leq \|f'\|_K$ for all $f' \sim f$. Two sequences $\{m_i\}$ and $\{n_i\}$ in $\mathbb{N}$ are *comparable* if there is a number $C > 0$ such that $C^{-1} \cdot m_i \leq n_i \leq C \cdot m_i$ for all i.

Corollary 7.6. *Let* $\{f_i\}$ *be a sequence in* $Hom(G, \mathbb{F})$*. Suppose that* $f_i' \sim f_i \sim f_i''$ *where* f_i' *is short and* f_i'' *is* K*-short. Then, the sequences* $\{|f_i'|\}$ *and* $\{\|f_i''\|_K\}$ *are comparable.* □

Definition 7.7. If ℓ is a leaf of a measured lamination Λ on a finite 2-complex K, then (conjugacy classes of) elements in the image of $\pi_1(\ell \subset K)$ are *carried by* ℓ. Suppose that Λ_i is a component of Λ. If Λ_i is simplicial (consists of a parallel family of compact leaves ℓ), then elements in the image of $\pi_1(\ell \subset K)$ are *carried by* Λ_i. If Λ_i is minimal and if N is a standard neighborhood[32] of Λ_i, then elements in the image of $\pi_1(N \subset K)$ are *carried by* Λ_i.

Definition 7.8. Let K be a finite 2-complex for G. Let $\{f_i\}$ be a sequence of short elements in $Hom(G, \mathbb{F})$ and let $\phi_i : \tilde{K} \to T_{\mathbb{F}}$ be an f_i-equivariant resolution. We say that the sequence $\{\phi_i\}$ is *short* if $\{\|\phi_i\|_K\}$ and $\{|f_i|\}$ are comparable.

Exercise 15. *Let* G *be freely indecomposable. In the setting of Definition 7.8, if* $\{\phi_i\}$ *is short then* $\Lambda = \lim \Lambda_{\phi_i}$ *has a leaf carrying non-trivial elements of* $Ker(T)$*.*

The idea is again that, if not, the induced GAD could be used to shorten. The next exercise, along the lines of Exercise 10, will be needed in the following lemma.[33]

Exercise 16. *Let* Δ *be a 1-edge GAD of a group* G *with a homomorphism* q *to a limit group* Γ*. Suppose:*

- *the vertex groups of* Δ *are non-abelian,*
- *the edge group of* Δ *is maximal abelian in each vertex group, and*

[32]see Theorem 3.6

[33]It is a consequence of Theorem 1.25, but since we are giving an alternate proof we cannot use this.

- *q is injective on vertex groups of* Δ.

Then, G is a limit group.

Lemma 7.9. *Let* Γ *be a limit group and let* $q : G \to \Gamma$ *be an epimorphism such that* $Hom(G, \mathbb{F}) = Hom(\Gamma, \mathbb{F})$. *If* $\alpha \in Mod(G)$ *then* α *induces an automorphism* α' *of* Γ *and* α' *is in* $Mod(\Gamma)$.

Proof. Since $\Gamma = RF(G)$, automorphisms of G induce automorphisms of Γ. Let Δ be a 1-edge splitting of G such that $\alpha \in Mod(\Delta)$. It is enough to check the lemma for α. We will check the case that $\Delta = A *_C B$ and that α is a Dehn twist by an element $c \in C$ and leave the other (similar) cases as exercises. We may assume that $q(A)$ and $q(B)$ are non-abelian for otherwise α' is trivial. Our goal is to successively modify q until it satisfies the conditions of Exercise 16.

First replace all edge and vertex groups by their q-images so that the second condition of the exercise holds. Always rename the result G. If the third condition does not hold, pull[34] the centralizers $Z_A(c)$ and $Z_B(c)$ across the edge. Iterate. It is not hard to show that the limiting GAD satisfies the conditions of the exercise. So, the modified G is a limit group. Since $Hom(G, \mathbb{F}) = Hom(\Gamma, \mathbb{F})$, we have that $G = \Gamma$ and $\alpha = \alpha'$. □

Alternate proof of the Main Proposition. Suppose that Γ is a generic limit group, $T \in \mathcal{T}'(\Gamma)$, and $\{f_i\}$ is a sequence of short elements of $Hom(\Gamma, \mathbb{F})$ such that $\lim T_{f_i} = T$. As before, our goal is to show that $Ker(T)$ is non-trivial, so suppose not. Recall that the action of Γ on T satisfies all the conclusions of Proposition 3.17.

Let $q : G \to \Gamma$ be an epimorphism such that G is fp and $Hom(G, \mathbb{F}) = Hom(\Gamma, \mathbb{F})$. By Lemma 7.9, elements of the sequence $\{f_i q\}$ are short. We may assume that all intermediate quotients $G \to G' \to \Gamma$ are freely indecomposable[35].

Choose a 2-complex K for G and a subsequence so that $\Lambda := \lim \Lambda_{f_i q}$ exists. For each component Λ_0 of Λ, perform one of the following moves to obtain a new finite laminated 2-complex for an fp quotient of G (that we will rename (K, Λ) and G). Let G_0 denote the subgroup of G carried by Λ_0.

[34]If A_0 is a subgroup of A, then the result of *pulling* A_0 across the edge is $A *_{\langle A_0, C\rangle} \langle A_0, B\rangle$, cf. moves of type IIA in [3].

[35]see [23]

1. If Λ_0 is minimal and if G_0 stabilizes a linear subtree of T, then enlarge $N(\Lambda_0)$ to a model for the action of $q(G_0)$ on T.

2. If Λ_0 is minimal and non-toral and if G_0 does not stabilize a linear subtree of T, then collapse all added annuli to their bases.

3. If Λ_0 is simplicial and G_0 stabilizes an arc of T, then attach 2-cells to leaves to replace G_0 by $q(G_0)$.

In each case, also modify the resolutions to obtain a short sequence on the new complex with induced laminations converging to Λ. The modified complex and resolutions contradict Exercise 15. Hence, $Ker(T)$ is non-trivial.

To finish, choose non-trivial $k_T \in Ker(T)$. As before, if $\{\mathcal{L}(\Gamma)\} \cup \{U(k_{T_i})\}$ is a finite cover for $\mathcal{T}'(\Gamma)$, then $\{\Gamma \to Ab(\Gamma)\} \cup \{\Gamma \to \Gamma/\langle\langle k_{T_i} \rangle\rangle\}$ is a factor set. □

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