The topology of $Out(F_n)$

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Introduction

$$Out(F_n) = Aut(F_n)/Inn(F_n)$$

Have epimorphism

$$Out(F_n) \to Out(\mathbb{Z}^n) = GL_n(\mathbb{Z})$$

and monomorphisms

$$MCG(S) \subset Out(F_n)$$

for surfaces S with $\pi_1(S) \cong F_n$.

Leitmotiv (Karen Vogtmann): $Out(F_n)$ satisfies a mix of properties, some inherited from mapping class groups, and others from arithmetic groups.

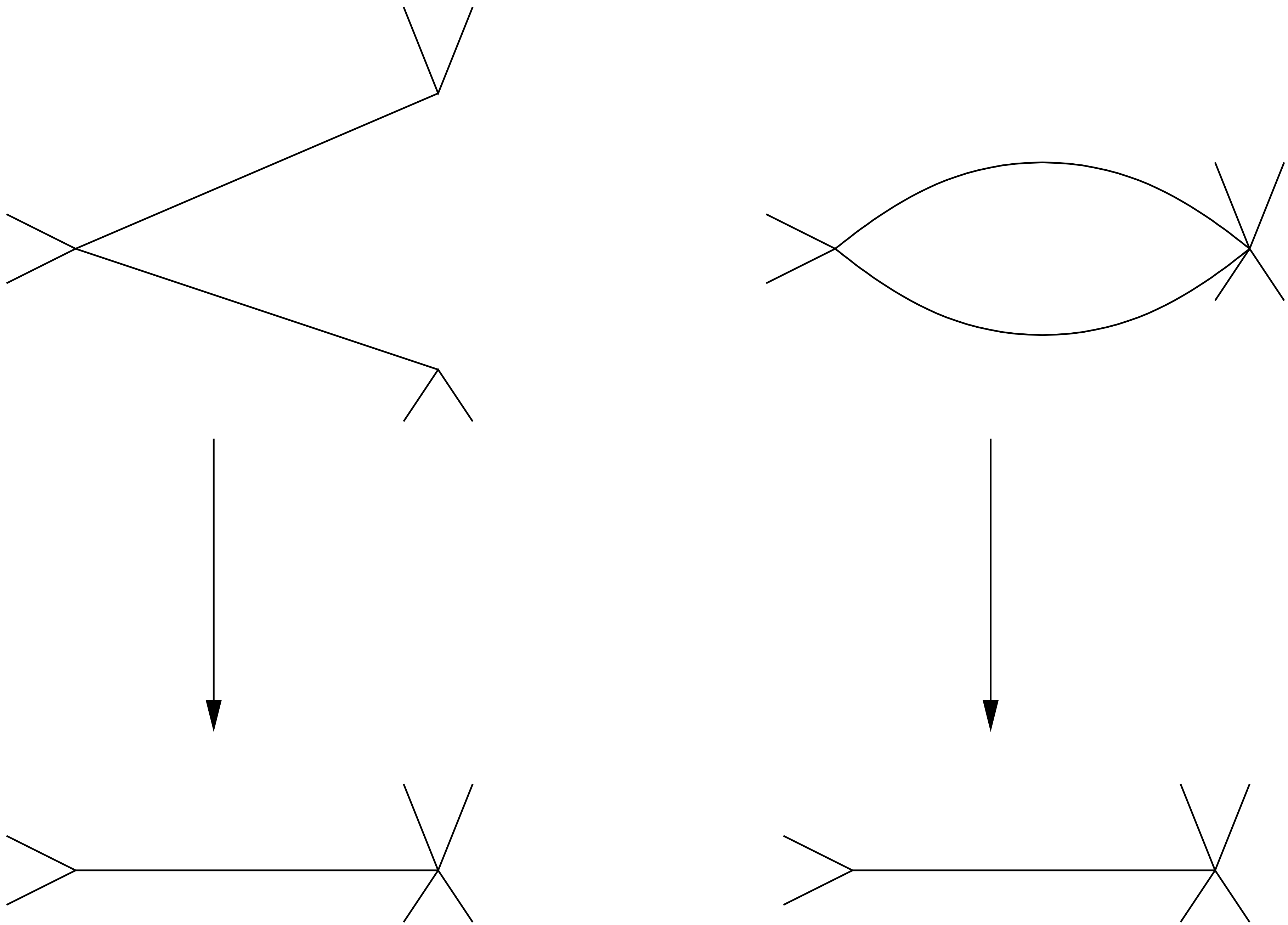
Mapping class groups	$Out(F_n)$	$GL_n(\mathbb{Z})$ **(arithmetic groups)**	**algebraic properties**
Teichmüller space	Culler-Vogtmann's Outer space	$GL_n(R)/O_n$ (symmetric spaces)	finiteness properties cohomological dimension
Thurston normal form	train track representative	Jordan normal form	growth rates fixed points (subgroups)
Harer's bordification	bordification of Outer space	Borel-Serre bordification	Bieri-Eckmann duality
measured laminations	**R**-trees	flag manifold (Furstenberg boundary)	Kolchin theorem Tits alternative
Harvey's curve complex	?	Tits building	rigidity

Stallings' Folds

Graph: 1-dimensional cell complex G

Simplicial map $f : G \to G'$: maps vertices to vertices and open 1-cells homeomorphically to open 1-cells.

Fold: surjective simplicial map that identifies two edges that share at least one vertex.



A fold is a homotopy equivalence unless the two edges share both pairs of endpoints and in that case the effect in π_1 is: killing a basis element.

Stallings (1983): A simplicial map $f : G \to G'$ between finite connected graphs can be factored as the composition

$$G = G_0 \to G_1 \to \cdots \to G_k \to G'$$

where each $G_i \to G_{i+1}$ is a fold and $G_k \to G'$ is locally injective (an immersion). Moreover, such a factorization can be found by a (fast) algorithm.

Applications: The following problems can be solved algorithmically (these were known classically, but folding method provides a simple unified argument). Let F be a free group with a fixed finite basis.

- Find a basis of the subgroup H generated by a given finite collection $h_1, \cdots, h_k$ of elements of F.

- Given $w \in F$, decide if $w \in < h_1, \cdots, h_k >$.

- Given $w \in F$, decide if w is conjugate into $< h_1, \cdots, h_k >$.

- Given a homomorphism $\phi : F \to F'$ between two free groups of finite rank, decide if ϕ is injective, surjective.

- Given finitely generated $H < F$ decide if it has finite index.

- Given two f.g. subgroups $H_1, H_2 < F$ compute $H_1 \cap H_2$ and also the collection of subgroups $H_1 \cap H_2^g$ where $g \in F$. In particular, is H_1 malnormal?

- Represent a given automorphism of F as the composition of generators of $Aut(F)$ of the following form:
  - Signed permutations: each a_i maps to a_i or to a_i^{-1}.
  - Change of maximal tree: $a_1 \mapsto a_1$ and for $i > 1$ a_i maps to one of $a_1^{\pm 1} a_i$ or to $a_i a_1^{\pm 1}$.

- Todd-Coxeter process

Culler-Vogtmann's Outer space

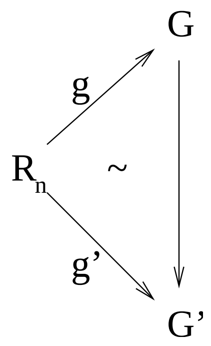
R_n: wedge of n circles. Fix an identification $\pi_1(R_n) \cong F_n$.

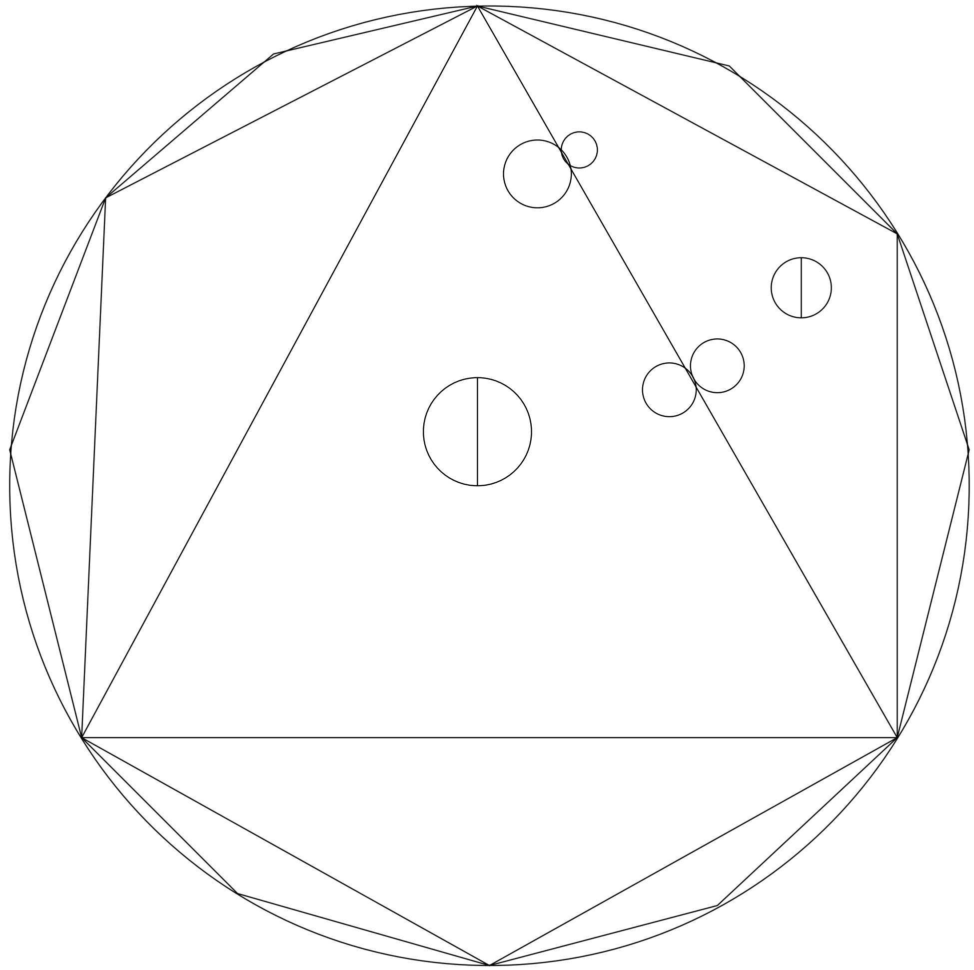
Any $\phi \in Out(F_n)$ can be thought of as a homotopy equivalence $R_n \to R_n$.

A *marked metric graph* is a pair (G, g) where

- G is a finite graph without vertices of valence 1 or 2.
- $g : R_n \to G$ is a homotopy equivalence (the *marking*).
- G is equipped with a path metric so that the sum of the lengths of all edges is 1.

Outer space X_n is the set of equivalence classes of marked metric graphs under the equivalence relation $(G, g) \sim (G', g')$ if there is an isometry $h : G \to G'$ such that gh and g' are homotopic.





If α is a loop in R_n we have the length function $l_\alpha : X_n \to \mathbb{R}$ where $l_\alpha(G,g)$ is the length of the immersed loop homotopic to $g(\alpha)$. The collection $\{l_\alpha\}$ as α ranges over all immersed loops in R_n defines an injection $X_n \to \mathbb{R}^\infty$ and the topology on X_n is defined so that this injection is an embedding. X_n naturally decomposes into open simplices obtained by varying edge-lengths on a fixed marked graph. The group $Out(F_n)$ acts on X_n on the right via

$$(G,g)\phi = (G,g\phi)$$

Culler-Vogtmann (1986): X_n is contractible and the action of $Out(F_n)$ is properly discontinuous (with finite point stabilizers). X_n equivariantly deformation retracts to a $(2n-3)$-dimensional complex.

Cor: The virtual cohomological dimension $\mathrm{vcd}(Out(F_n)) = 2n-3$.

Culler: Every finite subgroup of $Out(F_n)$ fixes a point of X_n.

Outer space can be equivariantly compactified (Culler-Morgan). Points at infinity are represented by actions of F_n on $\mathbb{R}$-trees.

Train tracks

Any $\phi \in Out(F_n)$ can be represented as a cellular map $f : G \to G$ on a marked graph G.

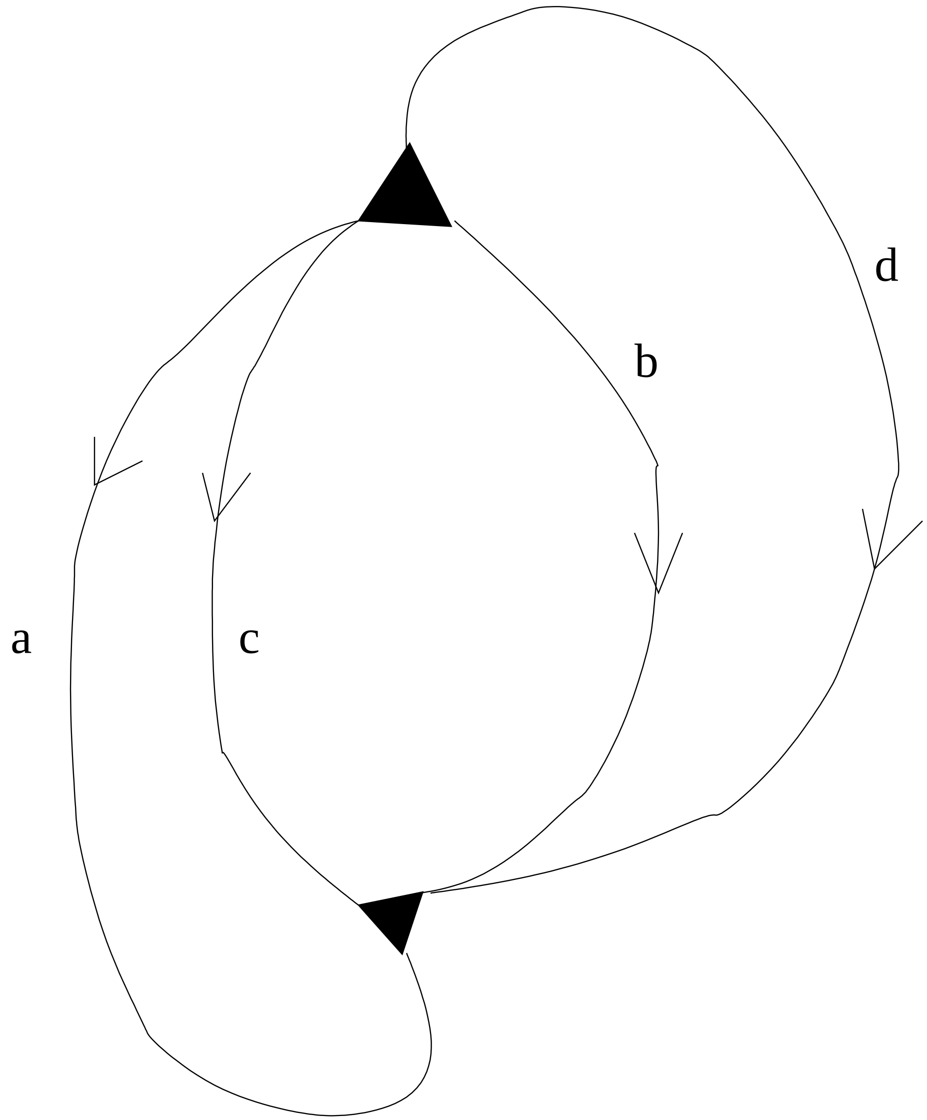
ϕ is *reducible* if there is a representative $f : G \to G$ where

- G has no vertices of valence 1 or 2, and
- there is a proper f-invariant subgraph of G with at least one non-contractible component.

Otherwise, ϕ is *irreducible*.

$f : G \to G$ is a *train track map* if for every $k > 0$ the map $f^k : G \to G$ is locally injective on every open 1-cell.

E.g., homeomorphisms are train track maps, so Culler's theorem guarantees that every $\phi \in Out(F_n)$ of finite order has a train track representative.



a−>B

b−>C

c−>D

d−>AbC .

B.-Handel (1992): Every irreducible outer automorphism ϕ can be represented as a train track map $f : G \to G$.

Perron-Frobenius: there is a metric on G such that f expands every edge, and also every legal path, by a uniform factor $\lambda \geq 1$.

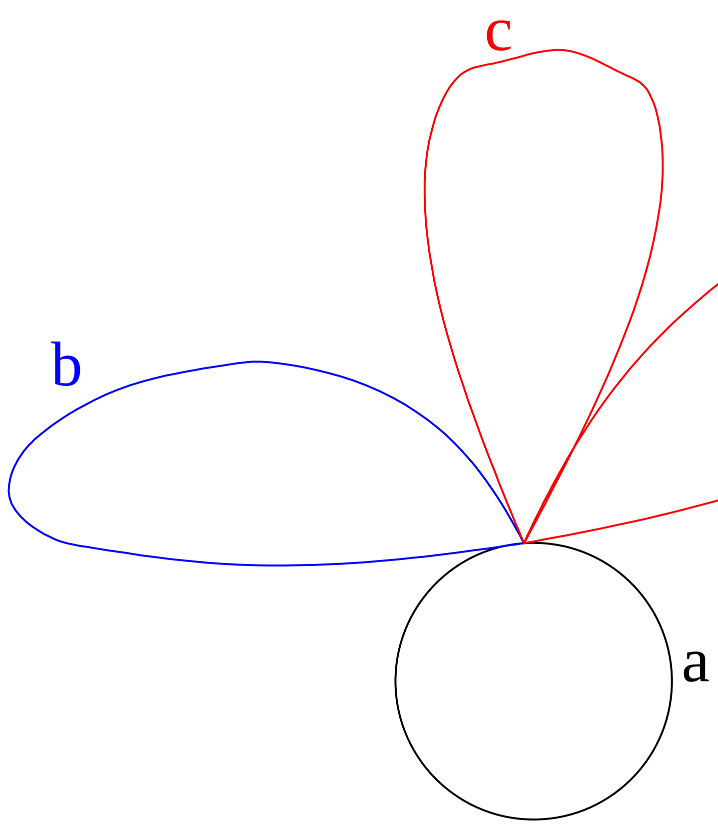
Cor: If ϕ is irreducible, then a conjugacy class γ is a either ϕ-periodic, or length $\phi^k(\gamma) \sim C\lambda^k$.

The proof uses a folding process that successively reduces the Perron-Frobenius number of the transition matrix until either a train track representative is found, or else a reduction of ϕ is discovered. This process is algorithmic.

Another application of train tracks is to fixed subgroups.

B.-Handel (1992) Let $\Phi : F_n \to F_n$ be an automorphism whose associated outer automorphism is irreducible. Then the fixed subgroup $Fix(\Phi)$ is trivial or cyclic. Without the irreducibility assumption, the rank of $Fix(\Phi)$ is at most n.

It was known earlier by the work of Gersten (1987) that $Fix(\Phi)$ has finite rank. The last sentence in the above theorem was conjectured by Peter Scott. Subsequent work by Dicks-Ventura (1993), Collins-Turner (1996), Ventura (1997), Martino-Ventura (2000) imposed further restrictions on a subgroup of F_n that occurs as the fixed subgroup of an automorphism. To analyze reducible automorphisms, a more general version of a train track map is required.



a->a

b ->a b

c -> c a b A B d

d -> d b c d

B.-Handel: Every automorphism of F_n admits a relative train track representative.

Automorphisms of F_n can be thought of as being built from building blocks (exponential and non-exponential) but the later stages are allowed to map over the previous stages. This makes the study of automorphisms of F_n more difficult (and interesting) than the study of surface homeomorphisms. Other non-surface phenomena (present in linear groups) are:

- stacking up non-exponential strata produces (nonlinear) polynomial growth,
- the growth rate of an automorphism is generally different from the growth rate of its inverse.

Related spaces and structures

Unfortunately, relative train track representatives are far from unique. As a replacement, one looks for canonical objects associated to automorphisms that can be computed using relative train tracks. There are 3 kinds of such objects, all stemming from the surface theory: laminations, $\mathbb{R}$-trees, and hierarchical decompositions of F_n (Sela).

Laminations. Laminations were used in the proof of the Tits alternative for $Out(F_n)$. To each automorphism one associates finitely many attracting laminations. Each consists of a collection of "leaves", i.e. biinfinite paths in the graph G. Roughly, they describe the limiting behavior of a sequence $f^i(\gamma)$. It is possible to identify the basin of attraction for each such lamination, and this makes ping-pong arguments possible in the presence of exponential growth.

There is an analog of Kolchin's theorem that says that finitely generated groups of polynomially growing automorphisms can simultaneously be realized as relative train track maps on the same graph (the classical Kolchin theorem says that a group of unipotent matrices can be conjugated to be upper triangular, or equivalently that it fixes a point in the flag manifold).The main step in the proof of the analog of Kolchin's theorem is to find an appropriate fixed $\mathbb{R}$-tree in the boundary of Outer space. This leads to the Tits alternative for $Out(F_n)$:

B.-Feighn-Handel (2000, to appear): Any subgroup $\mathcal{H}$ of $Out(F_n)$ either contains F_2 or is virtually solvable.

A companion theorem (B.-Feighn-Handel; Alibegović) is that solvable subgroups of $Out(F_n)$ are virtually abelian.

$\mathbb{R}$-**trees.** Points in the compactified Outer space are represented as F_n-actions on $\mathbb{R}$-trees. The Rips machine, which is used to understand individual actions, provides a new tool to be deployed to study $Out(F_n)$.

- computed the topological dimension of the boundary of Outer Space (Gaboriau-Levitt 1995)

- another proof of the fixed subgroup theorem (Sela 1996 and Gaboriau-Levitt-Lustig 1998)

- the action of $Out(F_n)$ on the boundary does not have dense orbits; however, there is a unique minimal closed invariant set (Guirardel 2000)

- automorphisms with irreducible powers have the standard north-south dynamics on the compactified Outer space (Levitt-Lustig 2002)

Cerf theory. (Hatcher-Vogtmann 1998)

Auter Space AX_n: similar to Outer Space, but graphs have a base vertex v

The degree of a graph: $2n -$ valence(v)

D_n^k: the subcomplex of AX_n consisting of graphs of degree $\leq k$.

Hatcher-Vogtmann:

- D_n^k is $(k-1)$-connected.
- $H_i(Aut(F_n))$ stabilizes when n is large.
- Explicit computations of rational homology for $i \leq 7$ (stably all are 0)

Bordification. The action of $Out(F_n)$ on Outer space X_n is not cocompact. B.-Feighn (2000) bordify X_n, i.e. equivariantly add ideal points so that the action on the new space BX_n is cocompact.

Ideal points are marked graphs with hierarchies of metrics.

A distinct advantage of BX_n over the spine of X_n (an equivariant deformation retract) is that the change in homotopy type of superlevel sets as the level decreases is very simple – via attaching of cells of a fixed dimension.

B.-Feighn (2000): BX_n and $Out(F_n)$ are $(2n-5)$-connected at infinity, and $Out(F_n)$ is a virtual duality group of dimension $2n-3$.

Mapping tori. $\phi : F_n \to F_n$ is an automorphism, $f : G \to G$ a representative.

The mapping torus $M(\phi) = \pi_1(G \times [0,1]/(x,1) \sim (f(x),0))$ plays the role analogous to 3-manifolds that fiber over the circle.

- $M(\phi)$ is coherent (Feighn-Handel 1999)
- When ϕ has no periodic conjugacy classes, $M(\phi)$ is a hyperbolic group (Brinkmann 2000).
- When ϕ has polynomial growth, $M(\phi)$ satisfies quadratic isoperimetric inequality (Macura 2000)
- If ϕ, ψ have polynomial growth and $M(\phi)$ is quasi-isometric to $M(\psi)$, then ϕ and ψ grow as polynomials of the same degree (Macura)

- Bridson and Groves announced that $M(\phi)$ satisfies quadratic isoperimetric inequality for all ϕ.

Geometry. biggest challenge in the field: find a good geometry that goes with $Out(F_n)$.

payoff (most likely) rigidity theorems for $Out(F_n)$

Mapping class groups and arithmetic groups act isometrically on spaces of nonpositive curvature.

The results to date for $Out(F_n)$ are negative.

- Outer space does not admit an equivariant piecewise Euclidean $CAT(0)$ metric (Bridson 1990)

- $Out(F_n)$ $(n > 2)$ is far from being a $CAT(0)$ group (Gersten 1994, Bridson-Vogtmann 1995)

An example of a likely rigidity theorem is that higher rank lattices in simple Lie groups do not embed into $Out(F_n)$. A possible strategy is to follow the proof in (B.-Fujiwara 2002) of the analogous fact (Kaimanovich-Farb-Mazur) for mapping class groups. The major missing piece of the puzzle is the replacement for Harvey's curve complex; a possible candidate is described by Hatcher-Vogtmann (1998).