In the following exercise assume we have a collection $\mathcal{Y}$ of metric spaces and projections $\pi_Z(X) \subset Z$ for distinct $X, Z \in \mathcal{Y}$ satisfying axioms (P1), (P2++), (P3) for some $\theta \geq 0$ (where we put $d_Y(X,Z) =: \text{diam}(\pi_Y(X) \cup \pi_Y(Z))$):

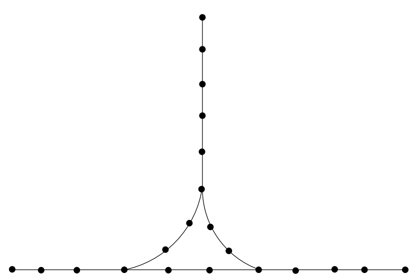
(P1) $\text{diam}\pi_Z(X) \leq \theta$,

(P2++) $d_Y(X,Z) > \theta$ implies $d_X(Y,Z) \leq \theta$,

(P3) $\{W \mid d_W(X,Z) > \theta\}$ is finite.

Also assume $K \geq 3\theta$.

1. (Triangles of standard paths) $\mathcal{Y}_K(X,Y) \cup \mathcal{Y}_K(Y,Z)$ contains all but at most two elements of $\mathcal{Y}(X,Z)$ and if there are two they are consecutive.



2. Let $n = |\mathcal{Y}_K(X,Z)| + 1$. Then

$$\lfloor \frac{n}{2} \rfloor + 1 \leq d_{\mathcal{P}_K(\mathcal{Y})}(X,Z) \leq n$$

Thus standard paths are quasi-geodesics and we have the "distance formula"

$$d_{\mathcal{P}_K(\mathcal{Y})}(X,Z) \asymp |\mathcal{Y}_K(X,Z)|$$

3. If $d_Y(X,Z)$ is sufficiently large (say $> 10K$) then every geodesic in $\mathcal{P}_K(\mathcal{Y})$ from X to Z passes through Y.

4. (Bounded Geodesic Image Theorem) There is M such that if $X_0, X_1, \cdots, X_n$ is a geodesic in $\mathcal{P}_K(\mathcal{Y})$ not passing through Y then $\text{diam}(\cup_i \pi_Y(X_i)) \leq M$.

Here are some hints.

1. Given $W \in \mathcal{Y}_K(X,Z)$ either $d_W(X,Y) > \theta$ or $d_W(Y,Z) > \theta$. If the former holds show that all $W' < W$ are in $\mathcal{Y}_K(X,Y)$.

2. If $d_{\mathcal{P}_K(\mathcal{Y})}(X,Z) = n$ need to show standard path has length $\leq 2n-1$. Induct on n. Choose Y on a geodesic between X and Z and draw the triangle of standard paths.

3. If $d_{\mathcal{P}_K(\mathcal{Y})}(X_i,Y) \geq 3$ for all i there is no progress in Y at all, but could have ≤ 5 X_i's with $d_{\mathcal{P}_K(\mathcal{Y})}(X_i,Y) \geq 2$ and then each time progress is $\leq K$.

4. Use that standard paths are quasi-geodesics. Break up the given geodesic into 3 subpaths. The middle path has bounded length and the other two have distance ≥ 3 from Y.