Math 6210 - Homework 1
Due at 4 PM on 9/1/05

We first recall the definition of the Riemann integral (on the interval $[0,1]$). A *partition*, $\mathcal{P}$, of the interval $[0,1]$ is a finite increasing sequence $x_0 < x_1 \cdots < x_n$ with $x_0 = 0$ and $x_1 = 1$. The partition $\mathcal{P}$ divides the interval $[0,1]$ into n subintervals $[x_{i-1}, x_i]$ of width $\Delta_i = x_i - x_{i-1}$. Given a function

$$f : [0,1] \longrightarrow \mathbb{R}$$

for each subinterval define

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

and

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

We then define the lower and upper Riemann sums by

$$L(f, \mathcal{P}) = \sum m_i \Delta_i$$

and

$$U(f, \mathcal{P}) = \sum M_i \Delta_i.$$

The lower and upper Riemann integrals are then

$$\underline{\int} f = \sup L(f, \mathcal{P})$$

and

$$\overline{\int} f = \inf U(f, \mathcal{P}).$$

We say that f is *Riemann integrable* if

$$\underline{\int} f = \overline{\int} f.$$

For a Riemann integrable function we write

$$\int f = \underline{\int} f.$$

1. Show that a continuous function on $[0,1]$ is Rieman integrable. You can do this however you like but below is an outline of a proof for which you can fiill in the details.

   (a) A partition $\mathcal{P}'$ is a *refinement* of $\mathcal{P}$ if $\mathcal{P}$ is contained in $\mathcal{P}'$ as a set. Show that

   $$L(f, \mathcal{P}) \leq L(\mathcal{P}')$$

   and

   $$U(f, \mathcal{P}) \geq U(\mathcal{P}').$$

(b) Show that for any two arbitrary partions $\mathcal{P}$ and $\mathcal{P}'$ we have

$$L(f,\mathcal{P}) \leq U(f,\mathcal{P}').$$

(Hint: Look at he common partion $\mathcal{P} \cup \mathcal{P}'$ of $\mathcal{P}$ and $\mathcal{P}'$.)

(c) Use the fact that a continuous function on a compact interval is uniformly continuous to show that for any $\epsilon > 0$ there exists a partition $\mathcal{P}$ with

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) \leq \epsilon.$$

(d) Finish the proof!

2. Let $\chi_{[a,b]}$ be the characteristic function of the interval $[a,b] \subseteq [0,1]$. Show that $\chi_{[a,b]}$ is Riemann integrable and that

$$\int_{[0,1]} \chi_{[a,b]} = b - a.$$

3. Let f be the function that is 1 on the rationals and 0 on the irrationals. Show that f is not Riemann integrable.

4. For any $\epsilon > 0$ show that there exists a closed subset A of the interval $[0,1]$ whose interior is empty but the Lesbesgue measure, $m(A)$, of A is $\geq 1 - \epsilon$. (Bonus: Show that there exists such an A with the $m(A) = 1 - \epsilon$.)