Math 6510 - Homework 3
Due in class on 9/26/13

1. Let M be a differentiable manifold. Recall that $v : \mathcal{C}^\infty(M) \to \mathbb{R}$ is a derivation at $x \in M$ if

   (a) $v(f + \lambda g) = v(f) + \lambda v(g)$ for all $f, g \in \mathcal{C}^\infty(M)$ and $\lambda \in \mathbb{R}$;

   (b) $v(fg) = f(x)v(g) + v(f)g(x)$.

   The space of all derivations at x is a vector space. Show that it is naturally isomorphic to T_xM. Here is an outline of how to do it.

   (a) If f is zero in a neighborhood of x use (b) to show that $v(f) = 0$. You can use the that for any open sets U and and V with $\bar{V} \subset U$ there exists a $\phi \in \mathcal{C}^\infty(M)$ with support in U and that is $\equiv 1$ on V. Use such a ϕ to decompose f into the product of two smooth functions that are zero at x.

   (b) If $f \equiv 1$ use (b) to show that $v(f) = 0$ and then use (a) to show that $v(f) = 0$ for all constant functions f.

   (c) Combine the previous two statements to show that if f is constant in a neighborhood of x then $v(f) = 0$.

   (d) If $f = g$ on a neighborhood of x show that $v(f) = v(g)$.

   (e) Reduce the statement to the following special case: If $M = \mathbb{R}^n$ and $x = 0$ then every derivation is of the form
   $$v(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}(0).$$
   Use the following calculus fact. If $f : \mathcal{C}^\infty(\mathbb{R}^n)$ with $f(0) = 0$ then in a neighborhood of 0
   $$f(x) = \sum_{i=1}^n x_i g_i(x)$$
   where the g_i are smooth functions with $\frac{\partial f}{\partial x_i}(0) = g_i(0)$.

2. Define an equivalence relation in $\mathbb{R}^2$ by $(x, y) \sim (x + n, (-1)^n y)$ and let $E = \mathbb{R}^2/\sim$. Note that the projection $\tilde{\pi} : \mathbb{R}^2 \to \mathbb{R}$ define by $\tilde{\pi}(x, y) = x$ descends to map $\pi : E \to S^1$. Show that this defines a vector bundle with fiber $\mathbb{R}$ and that this bundle is not a product.

3. Similar to the previous problem for points $(x, y) \in \mathbb{R} \times \mathbb{R}^2$ define $(x, y) \sim (x + n, (-1)^n y)$. As above this defines a vector bundle but this time with fiber $\mathbb{R}^2$. Show that this bundle is a product bundle.