Math 5510 Final - Part 1
Hand in these problems at the start of the exam. Handwritten solutions are OK.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be define by $f(x_1, x_2) = a_1 x_1^2 + a_2 x_2^2$ where $a_1, a_2 \in \mathbb{R}$ and let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be given by $F(x_1, x_2, x_3) = f(x_1, x_2) - x_3$.

1. Show that 0 is a regular value of F. (See the definition on the middle of p. 5 in the notes.)

2. Let $\Sigma = F^{-1}(\{0\})$ and show that the tangent space $T_{(0,0,0)}\Sigma$ is spanned by $(1, 0, 0)$ and $(0, 1, 0)$.

3. Show that the the shape operator $L\colon T_{(0,0,0)}\Sigma \to T_{(0,0,0)}\Sigma$ is given by the matrix

$$L = \begin{pmatrix} -2a_1 & 0 \\ 0 & -2a_2 \end{pmatrix}$$

   in terms of the basis from (2). Use this to calculate the Gauss and mean curvatures at $(0, 0, 0)$.

4. Let $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\alpha(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$. Show that α is a smooth parameterization of Σ. (See the definition at the top of p. 8 of the notes.)

5. Let E_1 and E_2 be the vector fields defined in Theorem 0.18 of the notes and g_{ij} the functions. Show that at $(0, 0, 0)$, E_1 and E_2 are an orthonormal basis of $T_{(0,0,0)}\Sigma$. Show that $g_{ii}(x_1, x_2) = 1 + 4a_i^2 x_i^2$ and $g_{12}(x_1, x_2) = g_{21}(x_1, x_2) = 4a_1 a_2 x_1 x_2$.

6. Use Theorem 0.18 to calculate $\langle D_{E_i} E_j, E_k\rangle$ and if either $x_1 = 0$ or $x_2 = 0$ show that $D_{E_1} E_1 = \frac{4a_1^2 x_1}{1+4a_1^2 x_1^2} E_1 + \frac{4a_1 a_2 x_2}{1+4a_2^2 x_2^2} E_2$, $D_{E_2} E_2 = \frac{4a_1 a_2 x_1}{1+4a_1^2 x_1^2} E_1 + \frac{4a_2^2 x_2}{1+4a_2^2 x_2^2} E_2$ and $D_{E_1} E_2 = D_{E_2} E_1 = 0$.

   (Remark: If v_1 and v_2 are orthogonal and a basis then any vector w can be written as $w = \langle w, v_1\rangle/\|v_1\|^2 + \langle w, v_2\rangle/\|v_2\|^2$. In particular if either $x_1 = 0$ or $x_2 = 0$ then E_1 and E_2 are orthogonal and we can use this to write the vector fields $D_{E_i} E_j$ in terms of E_1 and E_2.)