Notes on curves and surfaces

A *curve* is a smooth map from an interval I to $\mathbb{R}^n$. The interval I may be bounded or unbounded and may be open, closed or half-open. In these notes n will usually be 2 or 3 but many things will work for general n. The map is smooth if each coordinate functions (which will be a map from I to $\mathbb{R}$) is infinitely differentiable. (Although in most cases we only need at most two derivatives.)

Length: The first geometric concept we discuss is the length of a curve $\alpha\colon I \to \mathbb{R}^n$. To avoid issues of infinite length we assume that $I = [a, b]$ is a closed and bounded interval. The derivative of α is the map $\alpha'\colon I \to \mathbb{R}^n$ given by $\alpha'(t) = (\alpha_1'(t), \dots, \alpha_n'(t))$ where α_i are the coordinate functions. While the range of α' is formally the same as α we should think of a $\alpha'(t)$ as a vector in $\mathbb{R}^n$ rather than a point.

Let $\langle , \rangle$ be the standard inner product on $\mathbb{R}^n$ and define $\|v\| = \sqrt{\langle v, v\rangle}$. The quantity $\|v\|$ is the *(Euclidean) length* of the vector. We integrate this quantity to get the length of α:

$$L(\alpha) = \int_a^b \|\alpha'(t)\| dt$$

The length of a curve depends only its image not the specific choice of α. To make this statement precise we define a *reparameterization* of α to the be the pre-composition of α with a smooth, increasing homeomorphism $s\colon [c, d] \to [a, b]$. The reparameterization of α is then $\beta = \alpha \circ s$. The following lemma is a straightforward application of the substitution rule from calculus.

Lemma 0.1 *Let β be a reparameterization of α. Then $L(\beta) = L(\alpha)$.*

The curve α is a *unit-speed parameterization* if $\|\alpha'(t)\| = 1$. We then have:

Lemma 0.2 *Let $\alpha\colon I \to \mathbb{R}$ be a curve such that $\|\alpha'(t)\| \neq 0$ for all $t \in I$. Then α has unit-speed reparameterization.*

Proof. If $[a, b] \subset I$ define $\alpha[a, b]$ to be the restriction of α to $[a, b]$. Now fix some $a \in I$ and let

$$\sigma(t) = \begin{cases} L(\alpha[a,t]) & \text{if } t > a \\ 0 & \text{if } t = a \\ L(\alpha[t,a]) & \text{if } t < a \end{cases}$$

Then $\sigma'(t) = \|\alpha'(t)\|$ so if $\|\alpha'(t)\| \neq 0$ then σ has a smooth inverse $s\colon \sigma([a, b]) \to [a, b] \subset \mathbb{R}$. Applying the chain rule and the fundamental theorem of calculus we see that $\beta = \alpha \circ s$ is a unit-speed reparameterization. $\boxed{0.1}$

Geodesics: We can measure the distance between two points x and y in $\mathbb{R}^n$ by taking all curves that connect the two points and taking the infimum of the lengths. More precisely

$$d(x, y) = \inf\{L(\alpha) | \alpha\colon [a, b] \to \mathbb{R}^n, \alpha(a) = x, \alpha(b) = y\}.$$

As expected the shortest distance between two points is a line.

Proposition 0.3

$$d(x, y) = \|x - y\|$$

Proof. We first observe that if we define $\alpha\colon [0, 1] \to \mathbb{R}^n$ by $\alpha(t) = tx + (1-t)y$ then $\alpha(0) = x$, $\alpha(1) = y$ then $L(\alpha) = \|x - y\|$ and therefore $d(x, y) \leq \|x - y\|$.

To establish the inequality in the other direction we will show that for every path $\alpha\colon [a, b] \to \mathbb{R}^n$ we have $L(\alpha) \geq \|x - y\|$ and therefore $d(x, y) \geq \|x - y\|$. We'll first prove this when $n = 1$. Note that in this case the norm $\|\|$ is just the absolute value. For any $\alpha\colon [a, b] \to \mathbb{R}$ with $\alpha(a) = x$ and $\alpha(b) = y$ we have

$$L(\alpha) = \int_a^b |\alpha'(t)| dt \geq \left| \int_a^b \alpha'(t) dt \right| = |\alpha(b) - \alpha(a)| = |y - x|.$$

Note that the equality only occurs when $\alpha'(t) \neq 0$ for all $t \in [a, b]$.

Now we prove the inequality in general. Let $\alpha\colon [a, b] \to \mathbb{R}^n$ with $\alpha(a) = 0$ and $\alpha(b) = y$. Define $T\colon \mathbb{R}^n \to \mathbb{R}$ by $T(z) = \frac{\langle z, y-x \rangle}{\|y-x\|}$. The map T is smooth (in fact T is linear) so the composition $\beta = T \circ \alpha$ is smooth. Then $|\beta'(t)| \leq \|\alpha'(t)\|$ so $L(\beta) \leq L(\alpha)$. But β is a path from $\frac{\langle x, y-x \rangle}{\|y-x\|}$ to $\frac{\langle y, y-x \rangle}{\|y-x\|}$ so by the above calculation

$$L(\beta) \geq \left| \frac{\langle y, y - x \rangle}{\|y - x\|} - \frac{\langle x, y - x \rangle}{\|y - x\|} \right| = \|y - x\|.$$

$\boxed{0.3}$

Exercise: Show that $|\beta'(t)| \leq \|\alpha'(t)\|$.

Curvature: The next geometric quantity we want to measure is curvature. The motivating example is a circle of radius r where the curvature should be $1/r$. We will make a definition of curvature that gives this value for the circle. Our second motivation should be that the curvature should be the "2nd derivative" of the curve. Here we need to be careful as the 2nd derivative of a curve will be dependent on parameterization. In particular if the curve doesn't have a constant speed parameterization then the second derivative will have a component that measures this change in speed.

We'll start with a preliminary definition of curvature for unit-speed curves. If $\alpha\colon I \to \mathbb{R}^n$ is a smooth curve with a unit speed parameterization then *curvature* of α is the function $\kappa\colon I \to \mathbb{R}$ defined by $\kappa(t) = \|\alpha''(t)\|$.

The following lemma will be useful to calculate the curvature in general.

Lemma 0.4 *Let $\alpha\colon I \to \mathbb{R}^n$ be a unit-speed curve. Then $\langle \alpha'(t), \alpha''(t)\rangle = 0$.*

Proof. Differentiating the function $t \mapsto \langle \alpha'(t), \alpha'(t)\rangle$ we get $2\langle \alpha'(t), \alpha''(t)\rangle$. However since $\langle \alpha'(t), \alpha'(t)\rangle = \|\alpha'(t)\|^2 = 1$ is constant we have $\langle \alpha'(t), \alpha''(t)\rangle = 0$. $\boxed{0.4}$

To calculate the curvature in general we get the nicest formula if $n = 2$. Assume $\alpha\colon I \to \mathbb{R}^2$ and that $\|\alpha'(t)\| \neq 0$ for all $t \in I$. We would then like to define a normal vector $N\colon \alpha \to \mathbb{R}^2$ such that $\|N_\alpha(t)\| = 1$ and $\langle \alpha'(t), N_\alpha(t)\rangle = 0$. Unfortunately this only defines N up to sign. To define N uniquely we need to discuss orientation.

Orientation: Let V be a finite dimensional vector space. An *orientation* on V is an equivalence class of oriented bases where $\{v_1, \dots, v_n\} \sim \{w_1, \dots, w_n\}$ if there is a linear transformation $T\colon V \to V$ with $Tv_i = w_i$ and $\det T > 0$.

(You should reminder yourself of the definition of $\det T$. Given a basis for V one can define the determinant by writing T as a matrix in terms of the basis. Why is is the determinant independent of the choice of basis?)

Exercise: Show that $\sim$ is a equivalence relation and that there are exactly two equivalence classes. Show that $\{v_1, \dots, v_i, \dots, v_j, \dots, v_n\} \not\sim \{v_1, \dots, v_j, \dots, v_i, \dots, v_n\}$.

After fixing the standard orientation on $\mathbb{R}^2$ we assume that $\{\alpha'(t), N_\alpha(t)\}$ are an oriented basis.

Exercise: Show that $N\colon I \to \mathbb{R}^2$ is smooth.

For a unit-speed plane curve α we have $\kappa(t) = |\langle \alpha''(t), N_\alpha(t)\rangle|$.

Now let $\alpha\colon I \to \mathbb{R}^2$ be a plane curve and assume that $\|\alpha'(t)\| \neq 0$ but is not necessarily unit-speed. Let $s\colon J \to I$ be a smooth homeomorphism such that $\beta = \alpha \circ s$ is a unit-speed reparameterization of α. Note that for any reparameterization $N_\beta(t) = N_\alpha(s(t))$ is $s'(t) > 0$ and $N_\beta(t) = -N_\alpha(s(t))$ if $s'(t) < 0$. (Note that either $s'(t) > 0$ for all t or $s'(t) < 0$ for all t.)

Differentiating β twice we have

$$\beta''(t) = s'(t)^2 \alpha''(s(t)) + s''(t)\alpha'(t)$$

and therefore

$$\kappa(t) = |\langle \beta''(t), N_\beta(t)\rangle| = s'(t)^2\langle \alpha''(s(t)), N_\beta(t)\rangle + s''(t)\langle \alpha'(t), N_\beta(t)\rangle.$$

Since $N_\beta(t) = \pm N_\alpha(s(t))$, $\langle \alpha', N_\alpha\rangle = 0$ and $s'(t) = \frac{1}{\|\alpha'(s(t))\|}$ this becomes

$$\kappa(s) = \left| \frac{\langle \alpha''(s), N_\alpha(s)\rangle}{\|\alpha'(s)\|^2} \right|.$$

Curves as level sets: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. We will be interested in studying curves as sets of the form $\{x \in \mathbb{R}^2 | f(x) = c\}$ where $c \in \mathbb{R}$ is some constant.

For an arbitrary choice of c this set may not be a curve. For example the solutions to the equation $x^2 = 0$ is not a curve (at least near the origin in $\mathbb{R}^2$). The necessary condition is for c to be a *regular value* of f. To define regular values we need to define the derivative of a function from $\mathbb{R}^n \to \mathbb{R}^m$.

Let $U \subset \mathbb{R}^n$ be an open set and $f: U \to \mathbb{R}^n$. Then f can be thought of as m-functions of n-variables. The function f is smooth if each coordinate function is infinitely differentiable. Let $L(\mathbb{R}^n, \mathbb{R}^m)$ be linear maps from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$. Such a linear map is given by an $m \times n$ matrix so $L(\mathbb{R}^n, \mathbb{R}^m)$ is naturally identified with $\mathbb{R}^{nm}$. The derivative of f will be a map from U to $L(\mathbb{R}^n, \mathbb{R}^m)$ given by the matrix of partial derivatives. That is

$$f_*(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}.$$

The essential feature of the derivative is that it is the "best" linear approximation of f at x_0. More precisely for each $x \in U$ we can define a function $x \mapsto f(x) + f_*(x_0)(x - x_0)$. This is a continuous function that agrees with f at x. However, this would be true if we replace $f_*(x_0)$ with any linear function. What makes this the best linear approximation is that the function

$$v \mapsto \frac{\|f(x) - (f(x_0) + f_*(x_0)(x - x_0))\|}{\|x - x_0\|}$$

extends to a continuous function when $x = x_0$. (It is clearly continuous for $x \in U$ and $x \neq x_0$.) No other linear function has this property.

When $n = m$ we have the inverse function theorem.

Theorem 0.5 *Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^n$ a smooth function. Let $x \in U$ such that $f_*(x)$ is invertible. Then there exists an open neighborhood $V \subset f(U)$ of $f(x)$ in $\mathbb{R}^n$ and a smooth function $g: V \to U \subset \mathbb{R}^n$ such $f \circ g$ is the identity and $g_*(f(x)) = (f_*(x))^{-1}$.*

This is an extremely important theorem which we will not prove. (There are many references for the proof. The "hard" part is to show that the inverse exists and this uses the contraction mapping principle. The second part is to show that the inverse is smooth.)

Exercise: Assume that $n > m$ and that $T_1: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map of rank m. Show that there exists a linear map $T_2: \mathbb{R}^n \to \mathbb{R}^{n-m}$ so that the map $T: \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ defined by $Tv = (T_1 v, T_2 v)$ is invertible. Now let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$. Assume that $f_*(x_0)$ has rank m. Define $f_T: U \to \mathbb{R}^m$ by $f_T(x) = (f(x), T(x - x_0))$ where $T = T_2$ and $f_*(x_0) = T_1$. Show that $(f_T)_*(x_0) = (f_*(x_0), T)$. In particular $(f_T)_*(x_0)$ is invertible.

We will be mostly interested in maps $f\colon \mathbb{R}^n \to \mathbb{R}$. The inverse function theorem can be used to derive consequences about these maps. As an example of how this works start with a non-zero linear map $T\colon \mathbb{R}^n \to \mathbb{R}$. Then for all $x \in \mathbb{R}$, $T^{-1}(x)$ is a $n-1$-dimensional plane in $\mathbb{R}^n$. (In fancier terminology a *co-dimension 1 hyperplane.*) If $n = 2$ then a our "plane" is one-dimensional, a line. A line has a very simple parameterization as a smooth curve. Our next theorem gives a general condition for $f^{-1}(x)$ to be a curve when $f\colon U \to \mathbb{R}$ is a smooth map and $U \subset \mathbb{R}^2$ is open. To state it we first give another definition of a curve.

A subspace $\Gamma \subset \mathbb{R}^n$ is a smooth curve if for each $x \in \Gamma$ there is a smooth parameterized curve $\alpha\colon I \to \Gamma \subset \mathbb{R}^n$, $\alpha(I)$ contains a neighborhood of x in Γ and α is injective.

Let $f\colon U \to \mathbb{R}^m$ be smooth. Then $c \in \mathbb{R}^m$ is a *regular value* of f is for all $x \in f^{-1}(\{c\})$, $f_*(x)$ has rank m. Observe that

- If $m = 1$ the $f_*(x)$ has rank one if and only if $f_*(x) \neq 0$.
- If $m > n$ then the maximum rank of $f_*(x)$ is $n < m$ so $c \in \mathbb{R}^m$ is a regular value if and only if $f^{-1}(\{c\}) = \emptyset$.

Theorem 0.6 *Let $f\colon U \to \mathbb{R}$ be smooth and let $c \in \mathbb{R}$ be a regular value of f. Then $f^{-1}(\{c\})$ is a curve.*

Proof. To apply the Inverse Function Theorem we use the function $f_T\colon U \to \mathbb{R}^2$ from the exercise. Then $(f_T)_*(x_0)$ is invertible so there exists an neighborhood $V \subset \mathbb{R}^2$ of $f_T(x_0) = (f(x_0), 0)$ and a smooth function $g : V \to \mathbb{R}^2$ such that $g(V) \subset U$ and $f \circ g$ is the identity. After possible taking a sub-neighborhood we can assume that V is a ball centered at $f(x_0)$ of radius δ. We then define $\alpha\colon (-\delta, \delta) \to \mathbb{R}^2$ by $\alpha(t) = g(c, t)$. $\boxed{0.6}$

We would like to understand the geometry of $\Gamma = f^{-1}(\{c\})$ via the function f. Recall that the *gradient* of a function $f\colon U \to \mathbb{R}$ is $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$. Note that the linear map $f_*(x)$ is exactly the inner product with the gradient. That is $f_*(x)v = \langle \nabla f(x), v\rangle$. This leads to the following lemma.

Lemma 0.7 *Let $\alpha : I \to \mathbb{R}^2$ be a local parameterization of $\Gamma = f^{-1}(\{c\})$. Then $\langle \alpha'(t), \nabla f(\gamma(t))\rangle = 0$.*

Proof. The function $f \circ \gamma$ is constant so its derivative is zero. By the chain rule $0 = (f \circ \gamma)'(t) = f_*(\gamma(t))\gamma'(t) = \langle \nabla f(\gamma(t)), \gamma'(t)\rangle$. $\boxed{0.7}$

We would like to calculate the curvature of Γ in terms of ∇f. To do this we need to be able to differentiate "along" Γ. This is done with a directional derivative.

Let $f : U \to \mathbb{R}$ and $v \in \mathbb{R}^n$ where we think of v as a vector. Define $(\nabla_v f)(x) = \langle \nabla f(x), v\rangle$. The following lemma is more general version of Lemma 0.7.

Lemma 0.8 *Let $\alpha : I \to \mathbb{R}^n$ with $\alpha(t) = x$ and $\alpha'(t) = v$. Then $\nabla_v f(x) = (f \circ \alpha)'(t)$.*

Proof. The proof is exactly the same as Lemma 0.7. The only difference is the conclusion. Namely in the proof of Lemma 0.7 it is shown that $(f \circ \alpha)'(t) = \langle \nabla f(\alpha(t)), \alpha'(t) \rangle$. As $\nabla f(\alpha(t)) = \nabla f(x)$ and $\alpha'(t) = v$ we have $(\nabla_v f)(x) = \langle \nabla f(x), v \rangle = (f \circ \alpha)'(t)$ as desired. $\boxed{0.8}$

Note that one consequence of this lemma is that $\nabla_v f(x)$ only depends on the value of f on the image of the curve α.

We can apply the lemma to differentiate a vector field. In particular let $V : U \to \mathbb{R}^n$ be a smooth function. We want to think of V as a vector field so that its directional derivative will be a vector.

Lemma 0.9 *Let $\alpha : I \to U \subset \mathbb{R}^n$ be a smooth curve with $\alpha'(t) = v$. Then $(V \circ \alpha)'(t) = V_*(\alpha(t))v$.*

Again this is just the chain rule.

We now define $\nabla_v V(x) = V_*(x)v$. To calculate $\nabla_v V(x)$ we only need to now the values of V along a smooth curve α with $\alpha(t) = x$ and $\alpha'(t) = v$.

Exercise: Show the following:

1. $\nabla_v(V + W) = \nabla_v V + \nabla_v W$
2. $\nabla_v(fV) = (\nabla_v f)V + f\nabla_v V$
3. $\nabla_{v+w} V = \nabla_v V + \nabla_w V$
4. $\nabla_v \langle V, W \rangle = \langle \nabla_v V, W \rangle + \langle V, \nabla_v W \rangle$

Let $\alpha : I \to \mathbb{R}^2$ be a smooth curve. As we have seen the derivative of the tangent vector of a unit speed parameterization of the curve is the curvature. Furthermore, assuming an orientation of $\mathbb{R}^2$ there is unique normal vector at each point in the image of α. The derivative of this normal vector should similarly determine the curvature.

Exercise: Let $V(t)$ and $W(t)$ be vector valued functions. Then $\langle V(t), W(t) \rangle' = \langle V'(t), W(t) \rangle + \langle V(t), W'(t) \rangle$.

Given a curve Γ let N be the normal vector field on Γ. While N is not defined on open neighborhood we can still define $\nabla_v N$ if v is tangent to Γ. Given a particular smooth local parameterization $\alpha\colon I \to \mathbb{R}^2$ of Γ we then have $N_\alpha(t) = N(\alpha(t))$. With this set up we have the following lemma.

Lemma 0.10 *If $\alpha'(t) = v$ then $\langle \alpha''(t), N_\alpha(t) \rangle = -\langle (\nabla_v N(\alpha(t)), v \rangle$.*

Proof. We first observe that $N'_\alpha(t) = \nabla_{\alpha'(t)} N(\gamma(t)) = (\nabla_v N)(\gamma(t))$. The function $t \mapsto \langle \alpha'(t), N_\alpha(t)\rangle$ is identically equal to zero so differentiating it we see that

$$\begin{aligned} 0 &= \langle \alpha'(t), N_\alpha(t)\rangle' \\ &= \langle \alpha''(t), N_\alpha(t)\rangle + \langle \alpha'(t), N'_\alpha(t)\rangle \\ &= \langle \alpha''(t), N_\alpha(t)\rangle + \langle v, \nabla_v N(\alpha(t))\rangle. \end{aligned}$$

$\boxed{0.10}$

We can use this lemma to calculate the curvature of a curve $\Gamma = f^{-1}(\{c\})$ in terms of f. The central observation is that the vector field $\nabla f/\|\nabla f\|$ is defined everywhere that $\nabla f \neq 0$ and is also the normal vector at each point on the curve Γ.

The *Hessian* of a smooth function $f\colon U \to \mathbb{R}$ is the $n \times n$ matrix of second partial derivatives

$$Hf = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right).$$

The Hessian is most usefully thought of as a *bilinear* map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. If $v, w \in \mathbb{R}^n$ then if we multiply the matrix Hf by v on the left and w on the right we get a number.

Exercise: Let V be a finite dimensional vector space. Given a map $B\colon V \times V$ and $w \in V$ we define maps $B_w\colon V \to \mathbb{R}$ and $B^w\colon V \to \mathbb{R}$ by $B_w(v) = B(w, v)$ and $B^w(v) = B(v, w)$.Then B is *bilinear* if the maps B_w and B^w are linear. Show that for every bi-linear map there is a matrix M_B such that $B(v, w) = v^T M_B w$ where the right hand side is matrix multiplication and v and w are column vectors.

Lemma 0.11 *For all $v, w \in \mathbb{R}^n$, $\langle (\nabla_v \nabla f)(x), w\rangle = Hf(v, w)$. If $w \in (\nabla f(x))^\perp$ then $\langle (\nabla_v N)(x), w\rangle = Hf(v, w)/\|\nabla f\|$.*

Proof. Recall that $(\nabla_v g)(x) = g_*(x)v$. The inner product of two row vectors is matrix multiplication of the transpose of the first vector with the second vector so $\langle (\nabla_v g)(x), w\rangle = (g_*(x)v)^T w$. If $g = \nabla f$ then a straightforward calculation shows that $g_* = Hf$. This proves the first statement.

For the second we use the exercise to see that

$$\nabla_v(\nabla f/\|\nabla f\|) = (\nabla_v(1/\|\nabla f\|))\nabla f + (1/\|\nabla f\|)\nabla_v \nabla f.$$

The first term on the left is parallel to ∇f so the inner product with anything in $(\nabla f(x))^\perp$ is zero. Therefore

$$\langle \nabla_v N(x), w\rangle = (1/\|\nabla f\|)\langle (\nabla_v \nabla f)(x), w\rangle = Hf(v, w)/\|\nabla f\|.$$

$\boxed{0.11}$

Exercise: Let $f\colon U \to \mathbb{R}$ be a smooth map and $c \in \mathbb{R}$ a regular value. Calculate the curvature of $\Gamma = f^{-1}(\{c\})$ in terms of f.

Surfaces: A smooth, parameterized surface is a smooth map $\alpha\colon U \to \mathbb{R}^n$ where $U \subset \mathbb{R}^2$ is open, $\alpha_*(x)$ has rank two for all $x \in U$ and α is injective.

$\Sigma \subset \mathbb{R}^n$ is a smooth surface if each $x \in \Sigma$ has a neighborhood V (in the subspace topology) such that V is the image of a smooth parameterized surface.

Theorem 0.12 *Let $U \subset \mathbb{R}^3$ be open and $f\colon U \to \mathbb{R}$ a smooth map. If $c \in \mathbb{R}$ is a regular value then $\Sigma = f^{-1}(\{c\})$ is a smooth surface.*

Proof. The proof is the same strategy as the proof of Theorem 0.6. Fix $x_0 \in \Sigma$. We first find a function $f_T\colon U \to \mathbb{R}^3$ such that $(f_T)_*(x_0)$ is invertible using the exercise after the proof of the Inverse Function Theorem. Then there exists a neighborhood V of $f_T(x_0)$ (which we can assume is a ball centered at $f_T(x_0)$ of radius $\delta > 0$) and a smooth map $g\colon V \to U \subset \mathbb{R}^3$ such that $f \circ g$ is the identity. We then define $\alpha\colon B_d(0,\delta) \to \mathbb{R}^3$ by $\alpha(x_1, x_2) = g(c, x_1, x_2)$. $\boxed{0.12}$

We would like to define a notion of curvature for a surface. This is more complicated than the case of curves as there will be more than way to do it.

First we define the normal vector field to a smooth, parameterized surface in $\mathbb{R}^3$. Fix the standard orientation on $\mathbb{R}^3$. To define $N_\alpha(x_1, x_2)$ we define the curves $\alpha_1(t) = \alpha(t, x_2)$ and $\alpha_2(t) = \alpha(x_1, t)$. Let $v_1 = \alpha_1'(x_1)$ and $v_2 = \alpha_2'(x_2)$. Since $\alpha_*(x_1, x_2)$ has rank two these vector will be linearly independent. We then choose $N_\alpha(x_1, x_2)$ so that it is orthogonal to both v_1 and v_2 and $\{v_1, v_2, N_\alpha(x_1, x_2)\}$ is an oriented basis of $\mathbb{R}^3$.

For a general smooth surface Σ we can define a normal vector via the parameterizations defined near each point. In particular if $x \in \Sigma$ we choose a local parameterization $\alpha\colon U \to \Sigma \subset \mathbb{R}^3$ whose image contains x. We then define $N(x) = N_\alpha(x_1, x_2)$ where $\alpha(x_1, x_2) = x$. We need to make sure that this is well defined.

The *tangent space*, $T_x\Sigma$ of Σ at x is the subset of $\mathbb{R}^3$ given by

$$T_x\Sigma = \{v \in \mathbb{R}^3 | \text{there exists } \alpha\colon I \to \Sigma \text{ with } \alpha(t) = x, \alpha'(t) = v\}$$

We need to show that $T_x\Sigma$ is a linear subspace. We will only do this in the case when $\Sigma = f^{-1}(\{c\})$ for some regular value of c.

Proposition 0.13 $T_x\Sigma = (\nabla f(c))^\perp$.

Proof. First we show that $T_x\Sigma \subset (\nabla f(x))^\perp$. Let $\alpha\colon I \to \Sigma$ with $\alpha(t) = x$. Then exactly as in the proof of Lemma 0.4 we see that $\langle \nabla f(x), \alpha'(t)\rangle = (f \circ \alpha)'(t)$ but since the image of α is contained in Σ, $f \circ \alpha$ is constant equal to c and has derivative zero. Therefore $\langle \nabla f(x), \alpha'(t)\rangle = 0$.

The orthogonal complement, $(\nabla f(x))^\perp$ is a 2-dimensional subspace of $\mathbb{R}^3$. To show that $T_x\Sigma = (\nabla f(x))^\perp$ we need to show that $T_x\Sigma$ contains a 2-dimensional subspace. Let $\alpha\colon U \to \Sigma \subset \mathbb{R}^3$ be a smooth parameterization whose image contains x. That is $\alpha(x_1, x_2) = x$ for some $(x_1, x_2) \in U \subset \mathbb{R}^2$. Then for all vectors $v \in \mathbb{R}^2$ we define $\alpha_v(t) = (x_1, x_2) + tv$. Then $\alpha_v'(0) \in T_x\Sigma$. But $\alpha_v'(0) = \alpha_*(x_1, x_2)v$ so $T_x\Sigma$ contains the entire image of the linear map $\alpha_*(x_1, x_2)$. As α is a smooth parameterized surface, $\alpha_*(x_1, x_2)$ has rank 2, so its image is two dimensional. $\boxed{0.13}$

Exercise: Assume that $n < m$ and let $T_1\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map of rank n. Show that there exists a linear map $T_2\colon \mathbb{R}^{m-n} \to \mathbb{R}^m$ so that the linear map $T\colon \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$ defined by $T(v, w) = T_1 v + T_2 w$ is invertible. Now let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^m$ a smooth map. Assume that $f_*(x_0)$ has rank n. Define $f_T\colon U \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ by $f_T(x, v) = f(x) + Tv$ where $T = T_2$ and $f_*(x_0) = T_1$. Show that $(f_T)_*(x_0, 0)(v, w) = (f_*)(x_0)v + Tw$. In particular show that $(f_T)_*(x_0, 0)$ is invertible.

Exercise: Use the previous exercise to show that if $\Sigma \subset \mathbb{R}^3$ is a smooth surface for every $x \in \Sigma$ there is a neighborhood U of x in $\mathbb{R}^3$ and a smooth function $f\colon U \to \mathbb{R}$ with $c\ in\mathbb{R}$ a regular value and $f^{-1}(\{c\}) = \Sigma \cap U$. (Hint: Apply the previous exercise to a smooth local parameterization α of Σ near x.)

The tangent space is exactly the set of directions where we can differentiate the normal vector field. Observe that $\langle N, N\rangle$ is the constant function 1 on Σ so

$$\begin{aligned} 0 &= \nabla_v \langle N, N\rangle \\ &= \langle \nabla_v N, N\rangle + \langle N, \nabla_v N\rangle \\ &= 2\langle \nabla_v N, N\rangle \end{aligned}$$

and therefore $\nabla_v N(x) \in T_x\Sigma$. We then define the map

$$L\colon T_x\Sigma \to T_x\Sigma$$

by $Lv = -\nabla_v N(x)$. L_v is the *shape operator.* (Sometimes it is call the *Weingarten map.*) It describes the curvature of the surface.

Proposition 0.14 *Lv is linear. For all $v, w \in T_x\Sigma$ we have $\langle Lv, w\rangle = \langle v, Lw\rangle$.*

Proof. Linearity follows from a previous exercise. By Lemma 0.11, $\langle Lv, w\rangle = -\langle \nabla_v N(x), w\rangle = -Hf(v, w)$ and $\langle v, Lw\rangle = -\langle \nabla_w N(x), f\rangle = -Hf(w, v)$. But since $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j x_i}$ we have $Hf(v, w) = Hf(w, v)$. $\boxed{0.14}$

We originally thought of the curvature of a smooth curve as being a real number. A real number can also be thought of as linear map from the $\mathbb{R}$ to $\mathbb{R}$. This second idea generalizes to higher dimensions. In particular we can think of the shape operator as the curvature of the surface.

To make the relationship of the curvature of a surface with the curvature of a curve more concrete we can intersect Σ with a plane that is spanned by a normal vector and a tangent vector. This intersection, at least near x, will a curve and we can calculate its curvature.

Given $x \in \Sigma$ and $v \in T_x\Sigma$ with $\|v\| = 1$ we define map $P_v\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $P_v(x_1, x_2) = x + x_1 v + x_2 N(x)$. Let U_v be a neighborhood U_v of $0 \in \mathbb{R}^2$ such that $P_v(U_v) \subset U \subset \mathbb{R}^3$. The we define $f_v\colon U_v \to \mathbb{R}$ by $f \circ P_v$. If $f_v(y) = c$ then $(f_v)_*(y) = (f_*(P_v(y)))(P_v)_*(y)$. Note that $(P_v)_*(y)$ is the linear map given by the matrix $(v\ N(x))$ where v and $N(x)$ are viewed as column vectors. If $y = 0$ (and then $P_v(y) = x$) the kernel of $f_*(P_v(y)) = f_*(x)$ is the one-dimensional space spanned by $N(x)$ and it follows that $(f_v)_*(0) \neq 0$. The property of $(f_v)_*$ not being zero is an open condition so there is a neighborhood of 0 where $(f_v)_*$ is not zero.

Lemma 0.15 *The curve on* $\Gamma_v = (f_v)^{-1}(\{c\})$ *has curvature* $\langle Lv, v\rangle$ *at* $0 \in \Gamma_v$.

Proof. Let $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Note that $(P_v)_*(0)w = v$ so $(f_v)_*(0)w = 0$ and w is tangent to Γ_v at 0. Therefore the curvature of Γ_v is $\langle \nabla_w \nabla f_v(0), v\rangle / \|\nabla f_v(0)\| = Hf_v(0)(v,v)/\|\nabla f_v(0)\|$. By the chain rule $\|\nabla f_v(0)\| = \|f_*(P_v(0))(P_v)_*(0) = \|\nabla f(x)(v\ N(x))\| = \|(0\ \|\nabla f(x)\|)\| = \|\nabla f(x)\|$.

We also have $Hf_v(0)(a,b) = Hf(x)((P_v)_*(0)a, (P_v)_*(0)b)$ so $Hf_v(0)(w,w) = Hf(x)(v,v)$. Therefore $\langle \nabla_w \nabla f_v(0), v\rangle / \|\nabla f_v(0)\| = Hf(x)(v,v)/\|\nabla f(x)\| = \langle Lv, v\rangle$. $\boxed{0.15}$

Let V be a vector space with an inner product. Then a linear map $T : V \to V$ is *symmetric* if $\langle Tv, w\rangle = \langle v, Tw\rangle$. A key fact about symmetric linear maps is that they can be diagonalized by an orthogonal change of basis.

Proposition 0.16 *There exists an orthonormal basis of eigenvectors* $\{v_1, \dots, v_n\}$.

Proof. Let $S^{n-1} \subset V$ be the set of unit length vectors. We then define $f\colon S^{n-1} \to \mathbb{R}$ by $f(v) = \langle Tv, v\rangle$. This is a continuous function on a compact set and will realize it maximum at some $v_0 \in S^{n-1}$. Given $v \in v_1^\perp$ define $\alpha_v\colon \mathbb{R} \to S^{n-1}$ by $\alpha_v(t) = (\cos t)v_1 + (\sin t)v$ and define $f_v = f \circ \alpha_v$. The function f_v will have a maximum at $t = 0$ so

$$\begin{aligned} 0 = f_v'(0) &= \langle T\alpha_v'(0), \alpha_v(0)\rangle + \langle T\alpha_v(0), \alpha_v'(0)\rangle \\ &= 2\langle Tv_1, v\rangle \end{aligned}$$

so we have for all $v \in v_1^\perp$, $\langle Tv_1, v\rangle$ which implies that Tv_1 is parallel to v_1 and hence an eigenvalue of T.

As $\langle Tv_1, v\rangle = \langle v_1, Tv\rangle$ we also have $\langle v_1, Tv\rangle = 0$ for all $v \in v_1^\perp$ so T maps the linear subspace $v_1^\perp$ to itself. T restricted to this subspace will again be symmetric so we can inductively find an eigenvector $v_2 \in v_1^\perp$. $\boxed{0.16}$

We apply this to the shape operator L. The eigenvectors of L are the directions of *principal curvatures* while the eigenvalues are the principal curvatures. The determinant of L is the Gauss curvature and the trace divided by two is the mean curvature.

Intrinsic and extrinsic geometry of surfaces in $\mathbb{R}^3$: If Σ is a surface in $\mathbb{R}^3$ we can make Σ into a metric space in two distinct ways. First we can take the usual metric on $\mathbb{R}^3$ and restrict it to Σ. The second way is more involved. A *piecewise smooth path* on Σ is a piecewise smooth map

$$\alpha\colon [a,b] \to \Sigma \subset \mathbb{R}^3.$$

We can define the length of α as at the beginning of the notes. Then for $x, y \in \Sigma$ define $d_\Sigma(x,y)$ to be the infimum of $L(\alpha)$ for all path α with $\alpha(a) = x$ and $\alpha(b) = y$. This is the *intrinsic metric* on Σ. We will be interested in what properties of the surface can be detected purely from the intrinsic metric. The most important example of this is the Gauss curvature. If $f\colon \Sigma_1 \to \Sigma_2$ is an isometry between the two intrinsic metrics of surfaces then Gauss curvature of Σ_1 at $x \in \Sigma_1$ will be equal to Gauss curvature of Σ_2 at $f(x)$. This is not true for the mean curvature.

Local isometries: Let Σ_1 and Σ_2 be smooth surfaces in $\mathbb{R}^3$. We say that $f\colon \Sigma_1 \to \Sigma_2$ is a smooth map if there is a neighborhood U of Σ_1 such that f extends to a smooth map from U to $\mathbb{R}^3$.

Lemma 0.17 *Let U be a neighborhood of Σ_1 and assume that $f\colon U \to \mathbb{R}^3$ is a smooth map with $f(\Sigma_1) \subset \Sigma_2$. Then the image of $T_x\Sigma_1$ under that map $f_*(x)$ is contained in $T_{f(x)}\Sigma_2$.*

Proof. For each $v \in T_x\Sigma_1$ there is a smooth path $\alpha\colon I \to \Sigma_1$ with $\alpha(t) = x$ and $\alpha'(t) = v$. Then $\beta = f \circ \alpha$ is smooth path to Σ_2 with $\beta(t) = f(x)$ and by definition $\beta'(t) \in T_{f(x)}\Sigma_2$. By the chain rule $\beta'(t) = f_*(\alpha(t))\alpha'(t) = f_*(x)v$ and therefore $f_*(x)v \in T_{f(x)}\Sigma_2$. $\boxed{0.17}$

Exercise: Show that there is a well defined linear map $f_*(x)\colon T_x\Sigma_1 \to T_{f(x)}\Sigma_2$. In particular for any two extensions f_1 and f_2 of f to an open neighborhood of Σ_1 show that for all $v \in T_x\Sigma_1$ that $(f_1)_*(x)v = (f_2)_*(x)v$.

A smooth map $f\colon \Sigma_1 \to \Sigma_2$ is a *local isometry* if for all $x \in \Sigma_1$ and for all $v, w \in T_x\Sigma_1$ we have $\langle v, w\rangle = \langle f_*(x)v, f_*(x)w\rangle$.

The covariant derivative: We have already discussed how to take the (directional) derivative of a vector field in $\mathbb{R}^n$ and saw that if the vector field is only defined on a subset of $\mathbb{R}^n$ we can differentiate in the direction v at x as long as the vector field is defined along a smooth curve through x with tangent vector v. This allows to differentiate vector fields defined on a surface in directions tangent to the surface. However, this differentiation very much depends on how the surface is embedded. That is it is an extrinsic notion.

We now define the *covariant* derivative of a vector field and we will see that this derivative is intrinsic.

Let Σ be a smooth surface in $\mathbb{R}^3$ and $W\colon \Sigma \to \mathbb{R}^3$ a vector field. For $v \in T_x\Sigma$ we then define the covariant derivative by

$$D_v W(x) = \nabla_v W(x) - \langle \nabla_v W(x), N(x)\rangle N(x)$$

where N is the normal vector field. This is just the usual directional derivative projected orthogonally to the tangent space.

Exercise: The covariant derivative satisfies all of the properties of the directional derivative.

Let $\alpha\colon I \to \Sigma \subset \mathbb{R}^3$ be a smooth path. The tangent vector $\alpha'(t)$ of α will be a vector field defined along the image of α we we can differentiate it in each tangent direction. That is $(D_{\alpha'(t)}\alpha')(t)$ is a vector along α. We say that α is a *geodesic* if $(D_{\alpha'(t)}\alpha')(t) = 0$ for all $t \in I$.

Theorem 0.18 *Let $\alpha\colon \to \Sigma$ be a smooth local parameterization. And define a vector field E_1 on the image of α by $E_1(x) = \alpha_*(x_1,x_2)\begin{pmatrix}1\\0\end{pmatrix}$ where $\alpha(x_1,x_2) = x$. Similarly define E_2 by replacing $\begin{pmatrix}1\\0\end{pmatrix}$ with $\begin{pmatrix}0\\1\end{pmatrix}$. Then*

$$\langle D_{E_i}E_j, E_k\rangle = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right)$$

where $g_{ij}(x_1,x_2) = \langle E_i(\alpha(x_1,x_2)), E_j(\alpha(x_1,x_2))\rangle$.

Proof. We first observe that the vector fields E_i are tangent to Σ and hence orthogonal to the normal vector field N so

$$\begin{aligned}
\langle \nabla_{E_i}E_j, E_k\rangle &= \langle D_{E_i}E_j + \langle \nabla_{E_i}E_j, N\rangle N, E_k\rangle \\
&= \langle D_{E_i}E_j, E_k\rangle + \langle \nabla_{E_i}E_j, N\rangle\langle N, E_k\rangle \\
&= \langle D_{E_i}E_j, E_k\rangle.
\end{aligned}$$

Fix an $x = \alpha(x_1, x_2)$ in the image of α and let $\alpha_1(t) = \alpha(t, x_2)$. Similarly define α_2. We then have

$$\begin{aligned} \nabla_{E_i} E_j(x) &= (\alpha_j' \circ \alpha_i)'(t_i) \\ &= \frac{\partial^2 \alpha}{\partial x_i \partial x_j}(x_1, x_2). \end{aligned}$$

Since mixed partials are equal we have that $\nabla_{E_i} E_j = \nabla_{E_j} E_i$.

We now calculate

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x_k}(x) &= \left(\frac{\partial}{\partial x_k} \langle E_i \circ \alpha, E_j \circ \alpha \rangle \right)(x) \\ &= \left(\nabla_{E_k} \langle E_i, E_j \rangle \right)(x) \\ &= \langle \nabla_{E_k} E_i(x), E_j(x) \rangle + \langle E_i(x), \nabla_{E_k} E_j(x) \rangle. \end{aligned}$$

We can similarly compute $\frac{\partial g_{ik}}{\partial x_j}$ and $\frac{\partial g_{jk}}{\partial x_i}$. We then have

$$\begin{aligned} \frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} &= \langle \nabla_{E_j} E_i(x), E_k(x) \rangle + \langle E_i(x), \nabla_{E_j} E_k(x) \rangle + \\ & \quad \langle \nabla_{E_i} E_j(x), E_k(x) \rangle + \langle E_j(x), \nabla_{E_i} E_k(x) \rangle - \\ & \quad (\langle \nabla_{E_k} E_i(x), E_j(x) \rangle + \langle E_i(x), \nabla_{E_k} E_j(x) \rangle) \\ &= \langle \nabla_{E_i} E_j(x), E_k(x) \rangle + \langle E_i(x), \nabla_{E_k} E_j(x) \rangle + \\ & \quad \langle \nabla_{E_i} E_j(x), E_k(x) \rangle + \langle E_j(x), \nabla_{E_k} E_i(x) \rangle - \\ & \quad (\langle \nabla_{E_k} E_i(x), E_j(x) \rangle + \langle E_i(x), \nabla_{E_k} E_j(x) \rangle) \\ &= 2 \langle \nabla_{E_k} E_i(x), E_j(x) \rangle \\ &= 2 \langle D_{E_k} E_i(x), E_j(x) \rangle. \end{aligned}$$

$\boxed{0.18}$

Corollary 0.19 *Let $f \colon \Sigma_1 \to \Sigma_2$ be a local isometry. Let W_1 and W_2 be vector fields on Σ_1 and Σ_2 with $f_*(x) W_1(x) = W_2(x)$ for all $x \in \Sigma_2$. Let $z \in \Sigma_1$ and assume that $V_1 \in T_z \Sigma_1$ and $V_2 \in T_{f(z)} \Sigma_2$ with $f_*(z) V_1 = V_2$. Then $f_*(z)(D_{V_1} W_1(z)) = D_{V_2} W_2(f(z))$.*

Proof. Let α^1 be a smooth local parameterization of Σ_1 near x. Then $\alpha^2 = f \circ \alpha^1$ is a smooth local parameterization of Σ_2 near $f(x)$. As in the Theorem 0.18 we define the vector fields E_i^k and the functions g_{ij}^k. The central observations is that $g_{ij}^1 = g_{ij}^2$. Then Theorem 0.18 implies that

$$\langle D_{E_i^1} E_j^1, E_k^1 \rangle = \langle D_{E_i^2} E_j^2, E_k^2 \rangle.$$

The map f is a local isometry so we have that

$$\langle D_{E_i^1} E_j^1(z), E_k^1(z)\rangle = \langle f_*(z)(D_{E_i^1} E_j^1(z)), f_*(z)E_k^1(z)\rangle$$

and since $f_*(z)E_k^1(z) = E_k^2(f(z))$ we have that

$$\langle f_*(z)(D_{E_i^1} E_j^1(z)), E_k^2(f(z))\rangle = \langle D_{E_i^2} E_j^2(f(z)), E_k^2 f(z)\rangle.$$

A vector is determined by the value of its inner products with a basis so $f_*(z)(D_{E_i^1} E_j^1(z)) = D_{E_i^2} E_j^2(f(z))$ and this proves the corollary when $V_k = E_i^k$ and $W_k = E_j^k$.

We now write $V_1 = \lambda_1 E_1^1(z) + \lambda_2 E_2^1(z)$ as linear combination of the basis $\{E_1^1(z), E_2^1(z)\}$ of $T_z\Sigma_1$. Since $f_*(z)$ is linear and $V_2 = f_*(z)V_1$ we have that $V_2 = \lambda_1 f_*(z)E_1^1 + \lambda_2 f_*(z)E_2^1 = \lambda_1 E_1^2 + \lambda_2 E_2^2$. It follows that

$$\begin{aligned} f_*(z)(D_{V_1} E_i^1(z)) &= f_*(z)(\lambda_1 D_{E_1^1} E_i^1(z) + \lambda_2 D_{E_2^1} E_i^1(z)) \\ &= \lambda_1 D_{E_1^2} E_i^2(f(z)) + \lambda_2 D_{E_2^2} E_i^2(f(z)) \\ &= D_{\lambda_1 E_1^2 + \lambda_2 E_2^2} E_i^2(f(z)) \\ &= D_{V_2} E_i^2(f(z)). \end{aligned}$$

We now similarly write W_1 and W_2 as linearly combinations of the E_j^i. Note that to calculate $D_V W(z)$ we only need to know the value of V at z but we are differentiating W so we need to know W in an entire neighborhood of z. In particular we let λ_1 and λ_2 be functions on a neighborhood $f(z)$ in Σ_2 such that $W_2(x) = \lambda_1(x)E_1^2(x) + \lambda_2(x)E_2^2(x)$. Using the linearity of $f_*(x)$, as above, we see that $W_1(x) = (\lambda_1 \circ f)(x)E_1^1(x) + (\lambda_2 \circ f)(x)E_2^1(x)$.

We now compute

$$\begin{aligned} \nabla_{V_1}(\lambda_i \circ f)(z) &= (\lambda_i \circ f)_*(z)V_1 \\ &= (\lambda_i)_*(f(z))f_*(z)V_1 \\ &= (\lambda_i)_*(f(z))V_2 \\ &= \nabla_{V_2}\lambda_i(f(z)). \end{aligned}$$

Therefore we have

$$
\begin{aligned}
f_*(z)(D_{V_1}W_1(z)) &= f_*(z)\left(\sum_{i=1}^{2} D_{V_1}(\lambda_i \circ f)E_i^1(z)\right)\\
&= f_*(z)\left(\sum_{i=1}^{2}\big(\nabla_{V_1}(\lambda_i \circ f)(z)E_i^1(z) + (\lambda_i \circ f)(z)D_{V_1}E_i^1(z)\big)\right)\\
&= f_*(z)\left(\sum_{i=1}^{2}\big(\nabla_{V_2}\lambda_i(f(z))E_i^1(z) + (\lambda_i \circ f)(z)D_{V_1}E_i^1(z)\big)\right)\\
&= \sum_{i=1}^{2}\big(\nabla_{V_2}\lambda_i(f(z))E_i^1(z) + f_*(z)((\lambda_i \circ f)(z)D_{V_1}E_i^1(z))\big)\\
&= \sum_{i=1}^{2}\big(\nabla_{V_2}\lambda_i(f(z))E_i^1(z) + \lambda_i(f(z))D_{V_2}E_i^2(f(z))\big)\\
&= \sum_{i=1}^{2} D_{V_1}\lambda_i E_i^2(f(z))\\
&= D_{V_2}W_2(f(z)).
\end{aligned}
$$

$\boxed{0.19}$

The shape operator L is a linear map from $T_x\Sigma$ to itself. Given an orthonormal basis $\{E_1, E_2\}$ of $T_x\sigma$ we can write L as a matrix. Namely $L = \begin{pmatrix} \langle LE_1, E_1\rangle & \langle LE_1, E_2\rangle \\ \langle LE_2, E_1\rangle & \langle LE_2, E_2\rangle \end{pmatrix}$.

Theorem 0.20 *Assume that the vector fields E_i come from a a smooth local parameterization as in Theorem 0.18 and also assume that at x, $\{E_1(x), E_2(x)\}$ are an orthonormal basis for $T_x\Sigma$. Then the Gauss curvature of Σ at x is*

$$\langle (D_{E_2}D_{E_1} - D_{E_1}D_{E_2})E_1, E_2\rangle.$$

Proof. We make three observations before the main calculation:

1. As in the proof of Theorem 0.18

$$\langle D_{E_i}D_{E_j}Z, E_k\rangle = \langle \nabla_{E_i}D_{E_j}Z, E_k\rangle.$$

2. For a tangent vector field Z we have $\langle Z, N\rangle = 0$ so

$$
\begin{aligned}
0 &= \nabla_{E_i}\langle Z, N\rangle\\
&= \langle \nabla_{E_i}Z, N\rangle + \langle Z, \nabla_{E_i}N\rangle
\end{aligned}
$$

and therefore $\langle \nabla_{E_i}Z, N\rangle = -\langle E_i, \nabla_Z N\rangle$.

3. If α is the smooth local parameterization of Σ that determines the E_i then $\nabla_{E_i} Z = \frac{\partial}{\partial x_i}(Z \circ \alpha)$ so $\nabla_{E_j}\nabla_{E_i} Z = \frac{\partial^2}{\partial x_j \partial x_i}(Z \circ \alpha)$ and $\nabla_{E_j}\nabla_{E_i} = \nabla_{E_i}\nabla_{E_j}$.

Now we compute

$$\begin{aligned}
\langle D_{E_i} D_{E_j} Z, E_k \rangle &= \langle \nabla_{E_i} D_{E_j} Z, E_k \rangle \\
&= \langle \nabla_{E_i} (\nabla_{E_j} Z - \langle \nabla_{E_j} Z, N \rangle N), E_k \rangle \\
&= \langle \nabla_{E_i} \nabla_{E_j} Z - (\nabla_{E_j} \langle \nabla_{E_j} Z, N \rangle) N - \langle \nabla_{E_j} Z, N \rangle \nabla_{E_i} N, E_k \rangle \\
&= \langle \nabla_{E_i} \nabla_{E_j} Z, E_k \rangle + \langle Z, \nabla_{E_j} N \rangle \langle \nabla_{E_i} N, E_k \rangle
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle (D_{E_2} D_{E_1} - D_{E_1} D_{E_2}) E_1, E_2 \rangle &= \langle \nabla_{E_2} \nabla_{E_1} E_1, E_2 \rangle + \langle \nabla_{E_1} N, E_1 \rangle \langle \nabla_{E_2} N, E_2 \rangle - \\
&\quad \langle \nabla_{E_1} \nabla_{E_2} E_1, E_2 \rangle - \langle \nabla_{E_2} N, E_1 \rangle \langle \nabla_{E_1} N, E_2 \rangle \\
&= \langle L E_1, E_1 \rangle \langle L E_2, E_2 \rangle - \langle L E_2, E_1 \rangle \langle L E_1, E_2 \rangle
\end{aligned}$$

but this is exactly the determinant of the matrix representation of L in the basis $\{E_1, E_2\}$.

$\boxed{0.20}$

Exercise: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with $f(x) > 0$ for all $x \in \mathbb{R}$. We define $F\colon \mathbb{R}^3 \to \mathbb{R}$ by $F(x_1, x_2, x_3) = x_1^2 + x_2^2 - f(x_3)^2$.

1. Show that 0 is a regular value for F.

2. Let $\Sigma = F^{-1}(\{0\})$. If $x = (x_1, x_2, x_3) \in \Sigma$ show that the vectors $(-x_2, x_1, 0)$ and $(x_1 f'(x_3), x_2 f'(x_3), f(x_3))$ form a basis for $T_x\Sigma$.

3. The shape operator L is a linear map from $T_x\Sigma \to T_x\Sigma$. Write L as a matrix in terms of the basis from (2).

   Solution: The gradient of F is $\nabla F = (2x_1, 2x_2, -2(f(x_3)f'(x_3)))$ so on Σ we have $\|\nabla F\| = 2f(x_3)\sqrt{1 + f'(x_3)^2}$ since $x_1^2 + x_2^2 = f(x_3)^2$. The Hessian of F is given by the matrix

$$HF = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2(f(x_3)f''(x_3) + f'(x_3)^2) \end{pmatrix}$$

   By Lemma 0.11, $\langle Lv, w \rangle = \langle -\nabla_v N, w \rangle = -HF(v, w)/\|\nabla F\|$. Let v_1 and v_2 be the basis from (2). We then have

$$\langle L v_1, v_1 \rangle = -\frac{2x_1^2 + 2x_2^2}{2f(x_3)\sqrt{1 + f'(x_3)^2}} = -\frac{1}{\sqrt{1 + f'(x_3)^2}}$$

and

$$\begin{aligned}\langle Lv_2, v_2\rangle &= -\frac{2x_1^2 f'(x_3)^2 + 2x_2^2 f'(x_3)^2 - 2f(x_3)^2(f(x_3)f''(x_3) + f'(x_3)^2)}{2f(x_3)\sqrt{1+f'(x_3)^2}} \\ &= \frac{f(x_3)^2 f''(x_3)}{\sqrt{1+f'(x_3)^2}}\end{aligned}$$

Since $HF(v_1, v_2) = HF(v_2, v_1) = 0$ we have $\langle Lv_1, v_2\rangle = \langle Lv_2, v_1\rangle = 0$. We then write L as a matrix by

$$\begin{aligned}L &= \begin{pmatrix} \langle Lv_1, v_1\rangle/\|v_1\|^2 & \langle Lv_2, v_1\rangle/\|v_1\|^2 \\ \langle Lv_1, v_2\rangle/\|v_2\|^2 & \langle Lv_2, v_2\rangle/\|v_2\|^2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{f(x_3)^2\sqrt{1+f'(x_3)^2}} & 0 \\ 0 & \frac{f''(x_3)}{(1+f'(x_3)^2)^{3/2}} \end{pmatrix}\end{aligned}$$

4. Use the matrix representation of L to calculate both the Gauss and mean curvature of Σ at x.

   Solution: The Gauss curvature is

$$1\frac{f''(x_3)}{f(x_3)^2(1+f'(x_3)^2)^2}$$

   and the mean curvature is

$$\frac{f(x_3)^2 f''(x_3) - 1 - f'(x_3)^2}{f(x_3)^2(1+f'(x_3)^2)^{3/2}}.$$

5. Using the same function f define $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $\alpha(x_1, x_2) = (f(x_1)\cos x_2, f(x_1)\sin x_2, x_1)$. Show that $f(\mathbb{R}^2) = \Sigma$.

6. Show that $f_*(x_1, x_2)$ has rank two for all $(x_1, x_2) \in \mathbb{R}^2$.

7. Given α we can define vector fields E_1 and E_2 and functions g_{ij} as in Theorem 0.18. Show that $g_{11}(x_1, x_2) = f'(x_1)^2 + 1$ and $g_{22}(x_1, x_2) = f(x_1)^2$ and $g_{12} = g_{21} = 0$.

8. Show that $D_{E_1}E_1(x_1, x_2) = -D_{E_2}E_2(x_1, x_2) = \frac{f'(x_1)f''(x_1)}{1+f'(x_1)^2}E_1$ and $D_{E_1}E_2(x_1, x_2) = D_{E_2}E_1(x_1, x_2) = \frac{f'(x_1)}{f(x_1)}E_2$.

9. Use Theorem 0.20 to calculate the Gauss curvature. Your answer should agree with (4)!