Homework 3, Math 5510
September 22, 2015
Section 18: 3, 7(a)
Section 19: 2, 7, 8, 10
Section 20: 3, 4, 8

#21.7 Assume that $f_n \to f$ uniformly and fix $\epsilon > 0$ then there exists an $N > 0$ such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$. But

$$\bar{\rho}(f_n, f) = \sup_{x \in X}\{\min\{|f_n(x) - f(x)|, 1\}\} < \epsilon$$

so $f_n \in B_{\bar{\rho}}(f, \epsilon)$ so $f_n \to f$ in $\mathbb{R}^X$.

Now assume that $f_n \to f$ in $\mathbb{R}^X$. Then for for all $1 > \epsilon > 0$ there exists an $N > 0$ such that if $n > N$ then $f_n \in B_{\bar{\rho}}(f, \epsilon)$. In particular $\min\{|f_n(x) - f(x)|, 1\} < \epsilon$ for all $x \in X$. So if $n > N$ we have $|f_n(x) - f(x)| < \epsilon$ and $f_n \to f$ uniformly.

#21.8 Since $f_n \to f$ uniformly there exists an $N_1 > 0$ such that if $n > N_1$ then $d(f_n(x_n), f(x_n)) < \epsilon/2$. Since the f_n are continuous and the convergence is uniform by Theorem 21.6, f is continuous and $f(x_n) \to f(x)$ (since $x_n \to x$). Therefore there exists an $N_2 > 0$ such that if $n > N_2$, $d(f(x_n), f(x)) < \epsilon/2$. Applying the triangle inequality we have

$$d(f_n(x_n) - f(x)) \leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon$$

so $f_n(x_n) \to f(x)$.

#22.3 We first show the that for any continuous map $p : X \to Y$ if there is a continuous map $f : Y \to X$ such that $p \circ f$ is the identity map then p is quotient map. Let $U \subset Y$ be a subset with $p^{-1}(U)$ open. Then $f^{-1}(p^{-1}(U))$ is open since f is continuous but $f^{-1}(p^{-1}(U)) = (p \circ f)^{-1}(U) = U$ so this show that U most be open in Y. By assumption p is continuous so this shows that p is a quotient map.

We'll show that the quotient space is $\mathbb{R}$. Projection maps on product spaces are continuous so the restriction q of π_1 to A is also continuous. Define $f : \mathbb{R} \to A$ by $f(x) = (x, 0)$. Then $q \circ f$ is the identity so by the above paragraph q is a quotient map so the quotient space is $\mathbb{R}$.

To show that q is not a open take the open set $((-1, 1) \times (0, \infty)) \cap A = [0, 1) \times (0, \infty)$. The q-image of this open set is $[0, \infty)$ and is not open so q is not an open map.

The set $\{(x, y) \in \mathbb{R}^2 | y = 1/x\}$ is a closed subset of A but its q-image is $(0, \infty)$ is not closed so q is also not a closed map.

#22.4(a) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, y) = x + y^2$. Then $x_0 \times y_0 \sim x_1 \times y_1$ if and only if $g(x_0, y_0) = g(x_1, y_1)$ so if we can show that g is a quotient map then the quotient space will be $\mathbb{R}$. We follow the same approach as in Problem 22.3. Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (x, 0)$. Both f and g are continuous and $g \circ f$ is the identity so g is a quotient map and the quotient space is $\mathbb{R}$.

#22.4(b) We follow the same strategy and let $g : \mathbb{R}^2 \to [0, \infty)$ be defined by $g(x, y) = x^2 + y^2$ and let $f : [0, \infty) \to \mathbb{R}^2$ be defined by $f(x) = (x, 0)$. Both f and g are continuous and $g \circ f$ is the identity so g is a quotient space and $[0, \infty)$ is the quotient space.

#23.9 Let $Z = X \times Y - A \times B$ and let $C = \{(x, y) \in Z | z \notin A\}$ and $D = \{(x, y) \in Z | y \notin B\}$. Note that $Z = C \cup D$ since if $(x, y) \in Z$ then we must have either $x \notin A$ or $y \notin B$ (or possibly both). Let $(x_0, y_0) \in C$. Then any $(x_1, y_1) \in D$ is any the same connected component of Z since the sets $\{x_0\} \times Y$ and $X \times \{y_1\}$ are connected subsets of Z that have the point (x_0, y_1) in common so there union is connected. Similarly every point in C is in the same connected component as any point in D. This implies that $Z = C \cup D$ is connected.

#23.11 Assume that X is not connected and $A, B \subset X$ are a separation. Note that $p^{-1}(p(A)) = A$ since if $y \in P(A)$ then $p^{-1}(\{y\})$ must be entirely contained in A since otherwise $p^{-1}(\{y\}) \cap A$ and $p^{-1}(\{y\}) \cap B$ would be a non-trivial separation of the connected set $p^{-1}(\{y\})$. Similarly $p^{-1}(p(B)) = B$. Since the sets A and B are open and p is a quotient map this implies that $p(A)$ and $p(B)$ are open. They are also disjoint since $p^{-1}(p(A)) = A$ and $p^{-1}(p(B)) = B$ are disjoint. Therefore $p(A)$ and $p(B)$ are a non-trivial separation of Y, contradiction.

#23.12 Assume that $Y \cup A$ has a non-trivial separation C, D. Note that C and D are open in the subspace topology for $Y \cup A$. Since Y is connected it must be contained in C or D. Lets say it is C. Since D is disjoint from C, this implies that D is contained in A. In particular, since A is open in $X - Y$ is open in the subspace topology on $Y \cup A$ (and hence the subspace topology on A)

#24.3 Define $g : [0, 1] \to \mathbb{R}$ by $g(x) = f(x) - x$. Then $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$ so by the Intermediate Value Theorem there exists a $x \in [0, 1]$ such that $g(x) = 0$. But then $g(x) = f(x) - x = 0$ and $f(x) = x$ so x is the desired fix point.

For a counterexample let $f(x) = x/2 + 1/2$. Then $f(x) = x$ if and only if $x = 1$ so f doesn't have a fixed point on either $[0, 1)$ or $(0, 1)$.

#24.8(a) Yes. Let (x_α) and (y_α) be points in $\prod X_\alpha$. Then for each α there are paths $\gamma_\alpha : [0,1] \to X_\alpha$ with $\gamma_\alpha(0) = x_\alpha$ and $\gamma_\alpha(1) = y_\alpha$. Define a path $\gamma : [0,1] \in \prod X_\alpha$ by $\gamma(t) = (\gamma_\alpha(t))$. Since each coordinate function is continuous, γ is continuous with $\gamma(0) = (x_\alpha)$ and $\gamma(1) = (y_\alpha)$. Therefore γ is a path from (x_α) to (y_α).

#24.8(b) No. Take the topologists sine curve $A = \{(x,y) \in \mathbb{R}^2 | y = \sin(1/x) \text{ and } x > 0\} \subset \mathbb{R}^2$ is path connected but its closure is not.

#24.8(c) Yes. Let y_0 and y_1 be points in $f(X)$. Then there exists $x_i \in X$ with $f(x_i) = y_i$. The composition of a path from x_0 to x_1 with f is a path from y_0 to y_1.

#24.8(d) Yes. Let $x \in \cap A_\alpha$. Then for any $x_0, x_1 \in \cup A_\alpha$ there are paths γ_0 from x_0 to x and γ_1 from x to x_1. The concatenation of these paths is a path from x_0 to x_1 so the union is path connected.

#24.10 Fix $x_0 \in U$ and let A be the set of points $x \in U$ such that there is a path in U from x_0 to x. We will show that $A = U$ by showing that A is non-empty, open and closed. Clearly $x_0 \in A$ so A is non-empty. For all $x \in A$ there is a ball $B_d(x, \epsilon)$ that is contained in U. Balls are path connected so every $y \in B_d(x, \epsilon)$ is in the same path connected component as y and hence as x_0. Therefore $B_d(x, \epsilon) \subset A$ and A is open. If $x \in \bar{A}$ then every open neighborhood of x intersects A. As before we have a ball $B_d(x, \epsilon) \subset U$. Since this ball intersects A there is a path in U from x to a point in A and hence $x \in A$ and $A = \bar{A}$ is closed. Therefore A is non-empty, open and closed. Since U is connected, $A = U$.