Homework 4, Math 5510
September 22, 2015
Section 28: 1, 6
Section 29: 8
Section 33: 4, 5
Section 35: 3

#28.1 Let $x_n \in [0,1]^\omega$ have a 1 in the nth place and a 0 in all others. Then $\bar{\rho}(x_n, x_m) = 1$ if $n \neq m$ so every ball of radius 1/3 intersect at most one x_n. Therefore the set has not limit point.

#28.6 The map f is injective since if $f(x) = f(y)$ then $0 = d(f(x), f(y)) = d(x, y)$ which implies that $x = y$.

To see that f is surjective we follow the hint in the book. Assume that $a \notin f(X)$. Since X is compact, $f(X)$ is compact and hence closed (as X is a metric space and therefore Hausdorff). Since $X \backslash f(X)$ is open there exists an $\epsilon > 0$ such that the ϵ-ball centered at a is disjoint from $f(X)$. Now inductively define $x_n \in X$ by setting $x_1 = a$ and $x_{n+1} = f(x_n)$. For $n > 1$, $x_n \in f(X)$ so $d(a = x_1, x_n) \geq \epsilon$. Note that f^n, the n-fold composition of f will also be an isometry so if $n < m$ then $x_n = f^{n-1}(x_1)$ and $x_m = f^{n-1}(x_{m-n+1})$ so $d(x_n, x_m) = d(x_1, x_{m-n+1}) \geq \epsilon$. Therefore (as in 28.1) the set x_n has no limit point, contradicting the compactness of X. There f is surjective and, hence, bijective.

An isometry is continuous, and a continuous bijection of a compact space to itself is a homeomorphism.

#29.8 First we observe that $\mathbb{Z}_+$, with the subspace topology inherited from $\mathbb{R}$ has the discrete topology as does $\{1/n | n \in \mathbb{Z}_+\}$ so the two spaces are homeomorphic. Next we note that $\{0\} \cup \{1/n | n \in \mathbb{Z}_+\}$ is closed and bounded and therefore compact. Any set with the discrete topology is both locally compact and Hausdorff so by Theorem 29.1 $\{0\} \cup \{1/n | n \in \mathbb{Z}_+\}$ is the unique one-point compactification of $\mathbb{Z}_+$.

#33.4 First assume that $f : X \to [0,1]$ is continuous and $f^{-1}(\{0\}) = A$. Then $U_n = f^{-1}((-1/n, 1/n))$ are open and $\cap U_n = f^{-1}(\cap(-1/n, 1/n)) = f^{-1}(\{0\}) = A$ so A is a closed G_δ.

Now assume that there are open sets U_n such that $A = \cap U_n$. By Urysohn's lemma we can find continuous functions $f_n : X \to [0, 1/n^2]$ that f_n is 0 on A and $1/2^n$ on $X \backslash U_n$ and let $h_n = \sum_{k=1}^n f_k$. The sequence of functions h_n will converge uniformly to a function f since the sum $\sum 1/2^n$ converge and f will be non-negative and bounded above by1. Therefore h_n converges to a function f which will be continuous since the convergence is uniform. Then f is 0 on A and if $x \notin A$ then there exists an U_n such that $x \in U_n$ so $f(x) \geq 1/n^2$.

#33.5 First we show the the existence of the function f. By 33.4 we have functions $f_A, f_B : X \to [0,1]$ that vanish exactly on A and B, respectively. Then $f = \frac{f_A}{f_A+f_B}$ is the desired function.

If $f : X \to [0,1]$ is continuous and is 0 on exactly on A and 1 exactly on B then A is a closed G_δ by applying 33.4 to f and B is a closed G_δ by applying 33.4 to $1-f$.

#35.3 We first show that (1) $\Rightarrow$ (2). Assume there is a continuous map $f : X \to \mathbb{R}$ that is not bounded. Then f is a homeomorphism to its graph in $X \times \mathbb{R}$. There is a product metric on $X \times \mathbb{R}$ and the subspace metric on the graph of f will be an unbounded metric on X.

Next we show that (2) $\Rightarrow$ (3). If X is not limit point compact then there exists a countable set A that doesn't have a limit point. Since X is metrizable it is Hausdorff. For each $x \in A$ has a neighborhood U such that $U \cap A$ has finitely many points $x, x_1, \dots, x_n$. Let U_i be a neighborhood of x that is disjoint form x_i. Then the intersection of U with $\cap U_i$ will be an open neighborhood of x that only contains x in its intersection with A. In particular the subspace topology on A is the discrete topology. We can then find on surjective, continuous function from A to $\mathbb{Z}_+$ (since all functions from a space with the discrete topology are continuous) which extends to a continuous function on all of X by the Tietze extension theorem.

Finally, we show that (3) $\Rightarrow$ (1). Assume that X has an unbounded metric d. Then there exists a sequence of points x_n with $d(x_1, x_n) \to \infty$. But then x_n does not have a convergent subsequence, contradiction.