Let $f\colon \Omega \to \mathbb{C}$ be locally univalent (i.e. f' is nowhere zero). Then the *Schwarzian derivative* is

$$Sf(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

One can calculate all the usual properties of the Schwarzian directly from this formula but we will try to give a more motivated definition where the properties are more transparent.

Define $M_f\colon \Omega \to PSL_2\mathbb{C}$ to be the *osculating Möbius transformation* to f. That is $M_f(z)$ is the unique Möbius transformation that agrees with f to second order:

$$M_f(z)(z) = f(z), (M_f(z))'(z) = f'(z) \text{ and } (M_f(z))''(z) = f''(z).$$

The derivative

$$d(M_f)\colon T\Omega \to TPSL_2\mathbb{C}$$

is a map from tangent spaces. Each tangent space of $PSL_2\mathbb{C}$ is canonically identified with the Lie algebra, $sl_2\mathbb{C}$. Each tangent space of Ω is canonically identified with $\mathbb{C}$ which has canonical basis $\frac{\partial}{\partial z}$. Define a map

$$M'_f\colon \Omega \to sl_2\mathbb{C}$$

by

$$M'_f(z) = d(M_f)_z\left(\frac{\partial}{\partial z}\right).$$

1. Define a map $\pi\colon PSL_2\mathbb{C} \to \widehat{\mathbb{C}}$ by $\pi(\phi) = \phi(0)$. Show that this map is a submersion.

2. Let $\pi\colon M \to N$ be a submersion and $\tilde{v}$ a vector field on M with flow ϕ_t. Assume that there are diffeomorphisms $\psi_t\colon N \to N$ with $\pi \circ \phi_t = \psi_t \circ \pi$. Show that the pushforward $\pi_*\tilde{v}$ is well defined. That is show that if $\pi(x_0) = \pi(x_1)$ then $\pi_* v(x_0) = \pi_* v(x_1)$.

3. The Lie algebra $sl_2\mathbb{C}$ is the space of left-invariant vector fields of $PSL_2\mathbb{C}$. If v is a left invariant vector field show that the push-forward $\pi_* v$ is well defined.

4. A vector field is *conformal* if its flow is conformal. Show that $v = f\frac{\partial}{\partial z}$ is conformal if and only if f is holomorphic.

5. Show that a conformal vector field on all of $\widehat{\mathbb{C}}$ is of the form $(az^2 + bz + c)\frac{\partial}{\partial z}$.

6. Show that $(\pi_* v)(z) = (aw^2 + bw + c)\frac{\partial}{\partial w}$ for some $a, b, c \in \mathbb{C}$.

7. The Lie algebra $sl_2\mathbb{C}$ is the space of two-by-two complex, traceless matrices. Explicitly give the isomorphism between $sl_2\mathbb{C}$ and conformal vector fields on $\widehat{\mathbb{C}}$.

8. Let $\phi(z)$ be a holomorphic family in $PSL_2\mathbb{C}$. If we write $\phi(z)(w)$ as a power series, centered at z, we have

$$\phi(z)(w) = \sum_{n=0}^{\infty} a_n(z)(w-z)^n$$

where the $a_n(z)$ are holomorphic functions. If we differentiate with respect to z this becomes

$$\phi'(z)(w) = \sum_{n=0}^{\infty}(a_n'(z)(w-z)^n - na_n(z)(w-z)^{n-1}).$$

Assuming that $\phi(z_0)$ is the identity show that $\phi'(z_0)(w)$ is quadratic polynomial in w and conclude that

- $a_1(z_0) = 1$;
- $a_n(z_0) = 0$ if $n \neq 1$ (these first two only require that $\phi(z_0)$ is the identity);
- $a_n'(z_0) = 0$ if $n \geq 3$.

9. Assume that $M_f(z_0)$ is the identity and apply the above result to show that

$$M_f'(z_0) = \frac{f'''(z_0)}{2}(w-z_0)^2\frac{\partial}{\partial w}.$$

10. Given locally univalent maps $f\colon \Omega \to \mathbb{C}$ and $g\colon f(\Omega) \to \mathbb{C}$ show that

$$M_{g\circ f}(z) = M_g(f(z)) \circ M_f(z).$$

11. Define a map $PSL_2\mathbb{C} \times PSL_2\mathbb{C} \to PSL_2\mathbb{C}$ by $(\psi, \phi) \mapsto \psi \circ \phi$. Given $(v, w) \in sl_2\mathbb{C} \times sl_2\mathbb{C}$ (where we view v and w as conformal vector fields on $\widehat{\mathbb{C}}$) show that the derivative of this map at (ψ, ϕ) is given by $(v, w) \mapsto \phi^* v + w$.

12. We can write $M_{g\circ f}$ as a composition of maps

$$\Omega \to f(\Omega) \times PSL_2\mathbb{C} \to PSL_2\mathbb{C} \times PSL_2\mathbb{C} \to PSL_2\mathbb{C}$$

where the first map on the left is $z \mapsto (f(z), M_f(z))$, the second map is $(z, \phi) \mapsto (M_g(z), \phi)$ and the last map is the composition map from the previous problem. Applying the chain rule to this composition show that

$$M_{g\circ f}'(z) = f'(z)(M_f(z))^*(M_g'(f(z))) + M_f'(z).$$

13. Given $\phi \in PSL_2\mathbb{C}$ show that $M'_{\phi\circ f}(z) = M'_f(z)$.

14. Let $\phi = (M_f(z_0))^{-1}$ be the unique element in $PSL_2\mathbb{C}$ such that $M_{\phi\circ f}(z_0)$ is the identity and show that

$$M'_f(z_0) = \frac{(\phi \circ f)'''(z_0)}{2}(w - z_0)^2 \frac{\partial}{\partial w}.$$

15. Consider $((M_f(z_0))^{-1} \circ f)(z)$ as a function of z and let $Rf(z_0)$ be its third derivative evaluated at z_0. Show that

$$M'_f(z) = \frac{Rf(z)}{2}(w - z)^2 \frac{\partial}{\partial w}.$$

(This is just a rephrasing of the previous problem.)

16. Given $\phi \in PSL_2\mathbb{C}$ let $v(w) = (w - \phi(z))^2 \frac{\partial}{\partial w}$. Show that

$$(\phi^* v)(w) = \phi'(z)(w - z)^2 \frac{\partial}{\partial w}.$$

17. Show that

$$M'_{g\circ f}(z) = \left(\frac{f'(z)^2 Rg(f(z)) + Rf(z)}{2}\right)(w - z)^2 \frac{\partial}{\partial w}.$$

18. Show that $Sf(z) = Rf(z)$ and conclude that $S(g \circ f) = Sg(f(z))f'(z)^2 + Sf(z)$.