Introductory topics in Kleinian grops and hyperbolic 3-manifolds
Margulis lemma problems

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If M is a hyperbolic 3-manifold. Let $M^{\leq\epsilon}$ be the set of points in M whose injectivity radius is $< \epsilon$. We similarly define $M^{\leq\epsilon}$, $M^{\geq\epsilon}$ and $M^{>\epsilon}$.

For a hyperbolic 3-manifold the Margulis lemma takes the following form:

Theorem 0.1 *There exists a constant ε_3 such that each component of the $M^{<\varepsilon_3}$ is of the following type:*

1. *the open r-neighborhood of a simple closed geodesic of length $\leq \varepsilon_3$;*
2. *the quotient of a horoball by a parabolic group isomorphic to $\mathbb{Z}$;*
3. *the quotient of a horoball by a parabolic subgroup isomorphic to $\mathbb{Z}^2$.*

In case (1) the component is a *Margulis tube*. In case (2) the component is a *rank one cusp* and in case (3) the component is a *rank two cusp*.

1. Given a Euclidean structure on the torus and a closed geodesic on the torus there is a tube with boundary the Euclidean structure and meridian the closed geodesic.
2. Given a Euclidean structure there is a unique rank two cusp with boundary the given Euclidean structure.
3. Show that all rank one cusps are isometric.
4. For $\lambda \in \mathbb{C}$ define $\phi_\lambda(z) = \lambda z$. Given a point in $p \in \mathbb{H}^3$ calculate $d(p, \phi_\lambda(p))$. Find a value for λ with $|\lambda| \neq 1$ and a point p such that $d(p, \phi_\lambda^2(p)) < d(p, \phi_\lambda(p))$.
5. Let $M_\lambda = \mathbb{H}^3/[\phi_\lambda]$ be the quotient hyperbolic 3-manifold of the group of isometries generated by ϕ_λ (again with $|\lambda| \neq 1$). Find a λ such that injectivity radius is not a smooth function on M_λ.

6. (hard) Given any $\epsilon > 0$ and $D > 0$ show that there exists an $\epsilon' < \epsilon$ such that for any point $p \in M_\lambda$ of injectivity radius ϵ and a point $q \in M_\lambda$ of injectivity radius ϵ' we have $d(p, q) \geq D$.

7. Prove the same statement but replace M_λ with a rank one or rank two cusp. This is much easier.

8. Using the Margulis Lemma and (7) and (8) prove the statement for a general hyperbolic 3-manifold.