Introductory topics in Kleinian grops and hyperbolic 3-manifolds

Problems on hyperbolic surfaces

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Let X be a finite area hyperbolic surface homeomorphic to a fixed topological surface S.

1. Show that there exists a constant B depending only on S (or $\chi(S)$) such that for any $p \in X$ there is a non-trivial loop through p of length $< B$. Show that B can be chosen such $\pi \sinh^2(B/2) = \text{area}(X) = -2\pi\chi(S)$.

2. Let $X^{\geq \epsilon} \subset X$ be those points whose injectivity radius is $\geq \epsilon$. Show there exists a constant D depending only on ϵ and S such that each component of $X^{\geq \epsilon}$ has diameter $\leq D$.

3. Show there exists a constant C depending only on S such that every hyperbolic surface X has a pants decomposition of length $< C$.

4. Given $\epsilon > 0$ show there exists an L depending only on S such if γ is a closed geodesic with the property that every geodesic that intersects γ essentially has length $> L$ then $\ell_X(\gamma) \leq \epsilon$.

5. Given any L show that there is a constant W such that if γ is a closed geodesic of length $\leq L$ then γ has a collar of width W. This constant should only depend on L not the topology of the surface. Furthermore as $L \to 0$ the width $W \to \infty$. (This is a little tricky. A good reference is Buser's book on hyperbolic surfaces which can be found in the MSRI library.)

6. Given any L show that there exists an N such that if α and β are closed geodesics on X of length $\leq L$ then $i(\alpha, \beta) \leq N$.

7. Show that $d_{\mathcal{C}}(\alpha, \beta) \leq i(\alpha, \beta) + 1$.

8. Let Ω be an open subset of $\mathbb{C}$ and $\rho : \Omega \longrightarrow \mathbb{R}^+$ a smooth positive function. We define a Riemannian metric on Ω by taking the product of ρ with the Euclidean metric. Show that the Gaussian curvature of this metric is $-\rho^{-2}\Delta \log \rho$.

9. Use the previous problem to show that $\rho_D(z) = \frac{2}{1-|z|^2}$ and $\rho_H(z) = \frac{1}{\operatorname{Im} z}$ define hyperbolic metrics on the unit disk and upper half plane, respectively.

10. Prove Ahlfors' lemma: Let X be a hyperbolic surface, Y have a metric with curvature ≤ -1 and $f : X \longrightarrow Y$ a holomorphic map. Then f is 1-Lipschitz.

    Here is an outline of the proof. Let ρ be a function defining a conformal metric on the unit disk whose curvature is ≤ -1. In general we we need to assume allow ρ to have zeros. If you want to make things simpler you can assume that ρ is strictly positive. The main work is to show that $\rho \leq \rho_D$.

    (a) Let $\rho_r(z) = \frac{2}{r(r^2-|z|^2)}$ and let $u_r = \log \rho_r$. Let $v = \log \rho$. Show that

    $$\Delta(v - u_r) \geq e^{2v} - e^{2u_r}$$

    on the disk $|z| < R$. In particular, where $v > u$ the function $v - u_r$ is subharmonic and has no local maximums.

    (b) Show that $v \leq u_r$ on the disk $|z| < r$ for $r < 1$. (Hint: Apply the maximum principle to $v - u_r$ on the open set E where $v > u_r$ to show that E is the empty set.)

    (c) Show that $v \leq u_1$ and therefore $\rho \leq \rho_D$ on the disk $|z| < 1$.

    (d) To finish the proof let ρ be the pull back via $\tilde{f}$ of the lift of the Y-metric to $\tilde{Y}$. If f is a conformal map then ρ will have no zeros and (c) implies that that $\tilde{f}$ and therefore f are contractions. What happens if f is just holomorphic and f' has zeros?

11. Let X' be a hyperbolic cone surface with the same conformal structure as X and assume all cone angles are $> 2\pi$. Modify X' to a metric X'' of smooth negative curvature ≤ -1 such that X'' agrees with X' outside of an ϵ-neighborhood of the cone points and X'' has the same conformal structure as X.

12. Use the Ahlfors lemma and the previous problem to show that the conformal map from X to X' is 1-Lipschitz.