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SELF-BUMPING OF DEFORMATION SPACES OF HYPERBOLIC 3-MANIFOLDS

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Abstract

Let N be a hyperbolic 3-manifold and B a component of the interior of $AH(\pi_1(N))$, the space of marked hyperbolic 3-manifolds homotopy equivalent to N. We will give topological conditions on N sufficient to give $\rho \in \overline{B}$ such that for every sufficiently small neighbourhood V of ρ, $V \cap B$ is disconnected. This implies that $\overline{B}$ is not a manifold with boundary.

1. Introduction

In this paper we study aspects of the topology of deformation spaces of Kleinian groups. The basic object of study is $AH(\pi_1(N))$, the space of isometry classes of marked, complete hyperbolic 3-manifolds homotopy equivalent to N, where N is a compact, orientable, irreducible, atoroidal 3-manifold with boundary. The study of the global topology of $AH(\pi_1(N))$ was begun by Anderson, Canary and McCullough in [1] for the case in which N has incompressible boundary. They described necessary and sufficient criteria for two components of the interior of $AH(\pi_1(N))$ to "bump"; that is, to have intersecting closures. We address the question of when a component of the interior "self-bumps"; that is, if B denotes such a component, then when is there an element ρ in the closure of B such that for any sufficiently small neighborhood V of ρ in $AH(\pi_1(N))$ the set $V \cap B$ is disconnected? In this paper we will establish the following result:

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Theorem 4.5. *Let N be a compact, orientable, atoroidal, irreducible 3-manifold with boundary. Suppose that N contains an essential, boundary incompressible annulus whose core curve is not homotopic into a torus boundary component of ∂N. Let B be a component of the interior of $AH(\pi_1(N))$. Then there is a representation ρ in $\overline{B}$ such that for any sufficiently small neighborhood V of ρ in $AH(\pi_1(N))$ the set $V \cap B$ is disconnected.*

Note that this result applies even when N has compressible boundary.

In [12] McMullen, using projective structures and ideas of Anderson and Canary, proved Theorem 4.5 when N is an oriented I-bundle over a surface. Our techniques avoid the use of projective structures, and furthermore, even in the I-bundle case we will find bumping representations that are not detected with McMullen's methods. In a sequel, will we use the techniques developed here to study the topology of the space of projective structures with discrete holonomy.

We sketch the proof of Theorem 4.5 in the case where $N = S \times [0,1]$ is an I-bundle over a closed surface of genus ≥ 2. In this case the interior of $AH(\pi_1(N))$ consists of a single component of quasifuchsian structures on $M = \operatorname{int} N$, which is usually denoted $QF(S)$.

To construct the representation where bumping occurs we start with a hyperbolic structure on M with a curve removed. That is choose a simple closed curve, c, on S and let $\hat{M} = M - c \times \{1/2\}$. Then give $\hat{M}$ a geometrically finite hyperbolic structure, and denote this (complete, open) hyperbolic 3-manifold by $\hat{M}_\infty$. Now, $\pi_1(\hat{M})$ has many conjugacy classes of subgroups isomorphic to $\pi_1(S)$, for example $S \times \{1/4\}$ and $S \times \{3/4\}$ each define such a subgroup. However, to find our bumping representation we choose a non-standard subgroup of $\pi_1(\hat{M})$ by wrapping S around the removed curve (see Figure 1). Then the hyperbolic structure of $\hat{M}_\infty$ defines a representation of $\pi_1(\hat{M})$ and our choice of subgroup defines a representation, ρ_∞, of $\pi_1(S)$. The cover, M_∞, associated to this subgroup will be homeomorphic to M.

The next step is to construct an immersion, $f : N \longrightarrow \hat{M}$, in the homotopy class of ρ_∞, and then use f to pull back the hyperbolic structure, $\hat{M}_\infty$, to a hyperbolic structure, N_∞, on N. Then N_∞ will be a complete hyperbolic structure with boundary. Such a hyperbolic structure is not uniquely determined by the holonomy; one can create a different structure by simply perturbing the immersion f. The advantage of such structures is that given a small neighborhood V of ρ_∞, for each $\rho \in V$

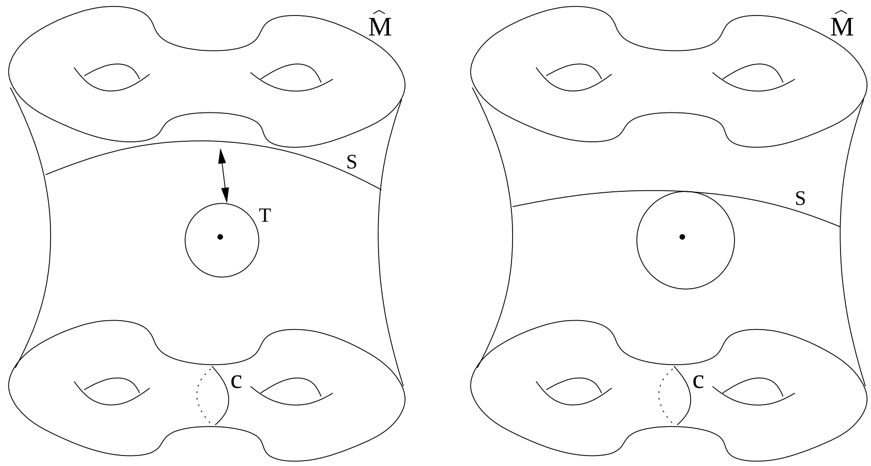

Figure 1: This schematic picture represents the immersion of S in $\hat{M}$. On the left, the torus T surrounds the removed curve $c \times \{1/2\}$ while the surface S is embedded. We form the immersion by cutting both S and T along a curve homotopic to c and then gluing the two surfaces together so that on the right, S wraps around the missing curve.

a general theorem allows us to construct a smoothly varying family of hyperbolic structures, N_ρ, with holonomy ρ, on the compact manifold N.

Now for each $\rho_\alpha \in AH(\pi_1(N))$, there is a hyperbolic 3-manifold, M_α, homeomorphic to M. Since N_α and M_α have the same holonomy there will be an isometric immersion, f_α, of N_α in M_α. If $\rho_\alpha \in V \cap QF(S)$ then c will have a geodesic representative, c_α, in M_α and there will be a canonical homeomorphism between $M_\alpha - c_\alpha$ and $\hat{M}$. Furthermore, geometric considerations will show that the image of f_α misses c_α so we can view f_α as a map to $\hat{M}$. In particular, we can compare the homotopy classes of the maps f_α in $\hat{M}$.

The cover M'_∞ of $\hat{M}_\infty$ corresponding ρ_∞ will be homeomorphic to M and f will lift to an embedding. The key to the proof is that we can find representations, ρ_0 and ρ_1 in $V \cap QF(S)$, such that the corresponding quasifuchsian manifolds will be geometrically close to $\hat{M}_\infty$ and M'_∞, respectively. In these new hyperbolic structures, N_0 will be immersed in M_0 while and N_1 will be embedded in M_1. In particular the maps f_0 and f_1 will not be homotopic in $\hat{M}$ and hence ρ_0 and ρ_1 cannot be in the same component of $V \cap QF(S)$.

It is worthwhile to compare this result with the bumping of distinct components examined in [2] and [1]. As mentioned above in [1], necessary and sufficient conditions are given for components to bump. We will not state them here, but at the very least we need a manifold with more topology than an I-bundle so that the interior of $AH(\pi_1(N))$ will have more than one component. The construction of the bumping representation is then very similar to the one above.

We first remove a suitably chosen simple closed curve c from $M = \operatorname{int} N$ to obtain an new manifold, $\hat{M}$. We then find a cover, M', of $\hat{M}$ that is homotopy equivalent, but not homeomorphic to M. A hyperbolic structure $\hat{M}_\infty$ on $\hat{M}$ induces a hyperbolic structure M'_∞ on M'. As above, there will be hyperbolic structures M_0 and M_1 that are geometrically close to $\hat{M}$ and M', respectively. In particular, M_0 and M_1 will not be homeomorphic and therefore cannot be in the same component of the interior of $AH(\pi_1(N))$.

The geometric phenomena is essentially the same for both bumping and self-bumping; it is the method of detection that is different. In both cases one finds a cover, M', of $\hat{M}$ that wraps around the removed curve c. For bumping M' will not be homeomorphic to M and this forces the nearby structures to be in different components of the interior of $AH(\pi_1(N))$. For self-bumping M' will be homeomorphic to M and the nearby structures will be in the same component of $AH(\pi_1(N))$. In this case, we need extra information, namely the homotopy class of f in $\hat{M}$, to detect the self-bumping.

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2. Preliminaries

2.1 Kleinian groups

A *Kleinian group* is a discrete, torsion free subgroup of the orientation preserving isometries of hyperbolic 3-space, $\mathbb{H}^3$. In the upper-half-space model of $\mathbb{H}^3$ the orientation-preserving isometries are identified with the group $PSL_2(\mathbb{C})$, so that a Kleinian group can be considered a discrete, torsion free subgroup of $PSL_2(\mathbb{C})$.

Let Γ be a Kleinian group and set M to be the quotient manifold $\mathbb{H}^3/\Gamma$. The *convex core* of M is the smallest convex submanifold of M whose inclusion in M is a homotopy equivalence. If the convex core has finite volume, and Γ is finitely generated then Γ is called *geometrically finite*. In addition, a geometrically finite Kleinian group is *minimally parabolic* if every maximal parabolic subgroup is of rank 2.

Let $R(\pi_1(N)) = \text{Hom}(\pi_1(N), PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})$ be the space of conjugacy classes of representations of $\pi_1(N)$ in $PSL_2(\mathbb{C})$ where N is a compact, orientable, atoroidal 3-manifold. The subset $AH(\pi_1(N)) \subset R(\pi_1(N))$ consists of the discrete, faithful representations of $\pi_1(N)$, modulo conjugacy. It is a result of Jørgensen [9] that $AH(\pi_1(N))$ is a closed subset of $R(\pi_1(N))$. By work of Marden [10] and Sullivan [14] the interior of $AH(\pi_1(N))$ is $MP(\pi_1(N))$, the minimally parabolic representations.

A representation $\rho \in AH(\pi_1(N))$ determines an oriented hyperbolic manifold $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$ along with a homotopy equivalence, $f_\rho : N \longrightarrow M_\rho$. In general $MP(\pi_1(N))$ will have many components. Two representations $\rho, \rho' \in MP(\pi_1(N))$ will be in the same component if and only if there exists a homeomorphism, $h : M_\rho \longrightarrow M_{\rho'}$, such that the maps $h \circ f_\rho$ and $f_{\rho'}$ are homotopic.

In this paper our interest is the topology of the closure of a single component, B. Since $AH(\pi_1(N))$ is determined only by the homotopy type of N, we can choose N such that ρ is in B if and only if M_ρ has a marking map that is an embedding.

2.2 Hyperbolic structures on compact manifolds

We also need to work with hyperbolic structures on the compact manifold N that may not extend to complete hyperbolic structures on an open manifold containing N. We let $\mathcal{H}(N)$ be the space of hyperbolic metrics on N. Given two hyperbolic metrics on N the identity map will be a biLipschitz map between the two metrics. Given a structure, $N' \in \mathcal{H}(N)$, a neighborhood $N'(\epsilon)$ of N' consists of those structures in $\mathcal{H}(N)$ for which the identity map from N' is a $(1+\epsilon)$-biLipschitz map. The $N'(\epsilon)$ are a basis of neighborhoods for N'.

Theorem 1.7.1 in [6] describes the local structure of a neighborhood of N'. We will need the following simple consequence of this theorem:

Theorem 2.1 ([6]). *The holonomy map $\mathcal{H}(N) \longrightarrow R(\pi_1(N))$ is locally onto. Furthermore, for any neighborhood V of N' there exists a neighborhood $U \subset V$, such that if N_0 and N_1 are hyperbolic structures in U with holonomy ρ_0 and ρ_1, respectively, and ρ_t, $0 \leq t \leq 1$, is a path in the image of U then there is a path N_t in U, where each N_t has holonomy ρ_t.*

2.3 Dehn surgery

Suppose that N is a compact, hyperbolizable 3-manifold and that ∂N contains at least one torus, T. Choose an oriented meridian and longitude for this torus such that elements of $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ are determined by a pair of integers, so that (m, l) denotes the curve which wraps m times around the meridian and l times around the longitude. Let (p, q) be a pair of relatively prime integers. Let $N(p, q)$ denote the result of performing (p, q)-Dehn filling on N along this torus; that is, there exists an embedding $d_{p,q} : N \longrightarrow N(p, q)$ such that $\overline{N(p, q) - d_{p,q}(N)}$ is a solid torus bounded by $d_{p,q}(T)$ and the image of the (p, q) curve on T is trivial in $N(p, q)$. Let γ denote the core curve of of the solid torus. Let M and $M(p, q)$ denote the interiors of N and $N(p, q)$, respectively. Suppose M_0 and $M_0(p, q)$ are complete hyperbolic structures on M and $M(p, q)$, respectively. We say that $M_0(p, q)$ is a *hyperbolic Dehn filling* of M_0 if $M_0(p, q) - d_{p,q}(M_0)$ contains the geodesic representative of γ. Note that a hyperbolic structure $M_0(p, q)$ may not be a hyperbolic Dehn filling of M_0 if γ is not isotopic to its geodesic representative. Also note that the holonomy representation ρ for $M_0(p, q)$ induces a non-faithful, holonomy representation $\rho_{p,q}$ for N via pre-composition with $(d_{p,q})_*$.

If N has k torus boundary components, we can Dehn fill each of them. Let relatively prime integers, (p_i, q_i), be the Dehn filling coefficients for the i-th torus and let $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \ldots; p_k, q_k)$. Then $N(\mathbf{p}, \mathbf{q})$ is the $(\mathbf{p}, \mathbf{q})$-Dehn filling of N.

The following theorem has an extensive history. The interested reader should also see [3], [15], [4], and [7].

Theorem 2.2 (The Hyperbolic Dehn Surgery Theorem ([5]). *Let N be a compact 3-manifold with k torus boundary components and let M denote its interior. Let M_ρ denote a minimally parabolic hyperbolic structure on the $\hat{M}$ with holonomy ρ. We then have the following:*

1. *There exist finite sets of relatively prime integers, P_i, $i = 1, \ldots, k$, such that for each collection of relatively prime pairs $(\mathbf{p}, \mathbf{q})$ with*

$(p_i, q_i) \notin P_i$ there exist a geometrically finite hyperbolic $(\mathbf{p}, \mathbf{q})$-Dehn filling of M_ρ.

2. *$\rho_{\mathbf{p},\mathbf{q}} \to \rho$ as $|\mathbf{p}, \mathbf{q}| \to \infty$, where $|\mathbf{p}, \mathbf{q}| = \min_i \{|p_i| + |q_i|\}$.*

3. *If X is the complement of a neighborhood of the cusps and $|\mathbf{p}, \mathbf{q}| > n$ then $d_{\mathbf{p},\mathbf{q}}|_X$ can be chosen to be K_n-biLipschitz with $K_n \to 1$ as $n \to \infty$.*

3. Wraps and twists

Let

$$X = [-1, 1] \times [-1, 1] \times S^1$$

and

$$\hat{X} = X - ([-\frac{1}{3}, \frac{1}{3}] \times [-\frac{1}{3}, \frac{1}{3}] \times S^1).$$

We begin by defining maps of the annulus,

$$A = [-1, 1] \times S^1$$

into $\hat{X} \subset X$. First we define $w : A \longrightarrow \hat{X}$ by

$$w(x, \theta) = \left(-\frac{1}{2} \sin(\pi x), \frac{1}{2} \cos(\pi x), \theta \right).$$

We next define a sequence of maps $w_n : A \longrightarrow \hat{X}$ for each $n > 0$. For each t and t' with $-1 \le t < t' \le 1$ we let $h_{t,t'} : ([t, t'] \times S^1) \longrightarrow A$ be a homeomorphism that satisfies the conditions, $h_{t,t'}(t, \theta) = (-1, \theta)$ and $h_{t,t'}(t', \theta) = (1, \theta)$. To define w_n we choose real numbers, $t_0, \dots, t_n$ with $-\frac{1}{3} = t_0 < t_1 < \dots < t_n = \frac{1}{3}$, and let

$$w_n(x, \theta) = \begin{cases} \left(\frac{3}{2}x + \frac{1}{2}, -\frac{1}{2}, \theta\right) & \text{if } -1 \le x < -\frac{1}{3} \\ w \circ h_{t_i, t_{i+1}} & \text{if } t_i \le x < t_{i+1} \\ \left(\frac{3}{2}x - \frac{1}{2}, -\frac{1}{2}, \theta\right) & \text{if } \frac{1}{3} \le x \le 1. \end{cases}$$

The map w_n wraps the annulus n times around the missing core of $\hat{X}$. For $n = 0$, we define w_0 by $w_0(x, \theta) = (x, -1/2, \theta)$.

Our next family of maps, $t_{n,m} : \hat{X} \longrightarrow \hat{X}$, are homeomorphisms which *Dehn twist* $\hat{X}$. They are defined by the following formula:

$$t_{n,m} = \begin{cases} (x, y, \theta) & \text{if } -1 \le x < -\frac{1}{3} \text{ or } \frac{1}{3} < x \le 1 \\ (x, y, \theta + 3n\pi(x + \frac{1}{3})) & \text{if } -\frac{1}{3} \le x \le \frac{1}{3} \text{ and } y > \frac{1}{3} \\ (x, y, \theta + 3m\pi(x + \frac{1}{3})) & \text{if } -\frac{1}{3} \le x \le \frac{1}{3} \text{ and } y < -\frac{1}{3}. \end{cases}$$

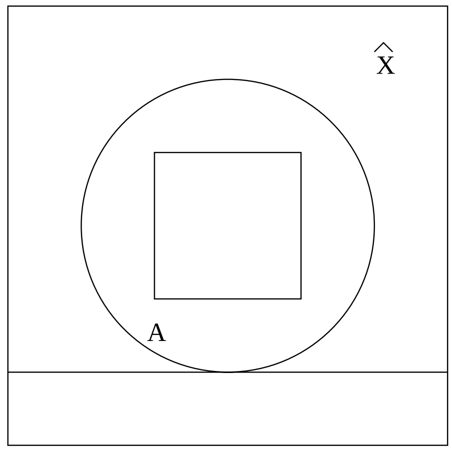

Figure 2: The image of A under the map w_1 in a cross section of $\hat{X}$.

Lemma 3.1. *The maps w_n and $t_{k(n+1),kn} \circ w_n$ are homotopic rel ∂A for any positive integer n and any integer k.*

Proof. Let $\hat{X}_{\frac{1}{3}} = ([-\frac{1}{3}, \frac{1}{3}] \times [-1,1] \times S^1) \cap \hat{X}$ denote the middle-third of $\hat{X}$; it has two components, the upper half and the lower half. The image of A under the map w_n intersects $\hat{X}_{\frac{1}{3}}$, so that $w_n^{-1}(w_n(A) \cap \hat{X}_{\frac{1}{3}})$ consists of $2n+1$ essential sub-annuli of A; n of the annuli map into the upper half of the middle third, while $n+1$ of the annuli map into the lower half. On the each of the $n+1$ annuli mapping into the lower half, $t_{k(n+1),kn}$ is a kn-Dehn twist, while on the n upper annuli $t_{k(n+1),kn}$ is a $-k(n+1)$-Dehn twist. Therefore the total effect of $t_{k(n+1),kn}$ is a $kn(n+1) - k(n+1)n = 0$-Dehn twist and w_n is homotopic to $w_n \circ t_{k(n+1),kn}$ rel ∂A (see Figure 3). q.e.d.

We now relate the maps $t_{n,m}$ to the Dehn filling of $\hat{X}$. As our coordinates for Dehn filling we choose the meridian to be the unique homotopy class that is trivial in X and the longitude to be the curve $\{\frac{1}{3}\} \times \{\frac{1}{3}\} \times S^1$. Recall the Dehn filling maps $d_{1,k} : \hat{X} \longrightarrow \hat{X}(1,k)$.

Lemma 3.2. *For each $t_{n,m}$, there exists a homeomorphism,*

$$h_{n,m} : \hat{X}(1, n-m) \longrightarrow X \supset \hat{X},$$

such that $t_{n,m} = h_{n,m} \circ d_{1,n-m}$ on $\hat{X}$. Furthermore the image of A under $h_{k(n+1),kn} \circ d_{1,n} \circ w_n$ is contained in $\hat{X}$ and $h_{k(n+1),kn} \circ d_{1,n} \circ w_n$

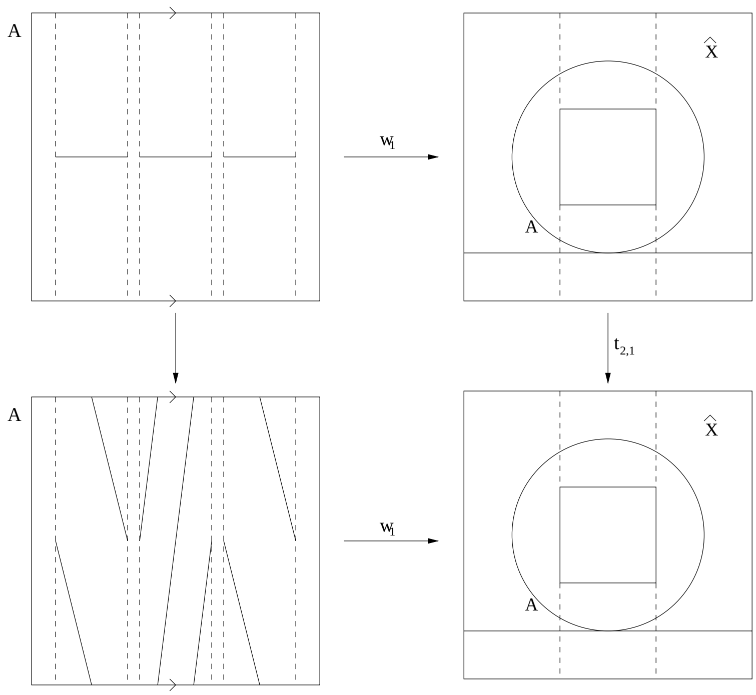

Figure 3: By identifying the top and bottom of the squares on the left we obtain (two copies of) the annulus A. The intersection of the image A under the map w_1, is the three dashed annuli. The effect of $t_{2,1}$ on A, is two Dehn twists on the center annuli and a single Dehn twist in the opposite direction on the two outside annuli. As we see from the picture in the lower left, the net effect on A is a map that is homotopic to the identity.

is homotopic in $\hat{X}$ *to* w_n *rel* ∂A *while* $h_{k(n+1),kn} \circ d_{1,n} \circ w_n$ *is homotopic to* w_0 *in* X.

Proof. On the image of $\hat{X}$ in $\hat{X}(1, n-m)$ define $h_{n,m} = t_{n,m} \circ d^{-1}_{1,n-m}$. Since the map $t_{n,m}$ takes the $(1, n-m)$-curve to the $(1,0)$-curve $h_{n,m}$ can be extend to a homeomorphism.

The map $h_{k(n+1),kn} \circ d_{1,n} \circ w_n$ is homotopic to w_n by Lemma 3.1. Since w_n is homotopic to w_0 in X, $h_{k(n+1),kn} \circ d_{1,n} \circ w_n$ is also homotopic to w_0 in X. q.e.d.

Let N be a compact, irreducible, and atoroidal 3-manifold with boundary that contains an essential, boundary incompressible annulus and let M be the interior of N. Furthermore assume that the core curve of the annulus is primitive and is not homotopic to a torus component of ∂N. We will use the wrapping maps, w_n, to define a class of immersion of N into M.

Let
$$\partial_0 X = [-1,1] \times \{-1,1\} \times S^1 \subset X$$
and
$$\partial_1 X = \{1\} \times [-1,1] \times S^1 \subset X.$$
Then there is a pairwise embedding of (X, ∂_0) in $(N, \partial N)$ such that $\partial_1 X$ is the essential boundary incompressible annulus given in the definition of N. We will abuse notation and refer to X as a submanifold of N. Identify A with the lower half of $\partial_0 X$; that is, the annulus $[-1,1] \times \{-1\} \times S^1$. Let $c = \{0\} \times \{0\} \times S^1$ be the core curve of X and let $\hat{M} = M - c$.

For each integer $n \geq 0$ we define an immersion
$$s_n : N \longrightarrow \hat{M} \subset M \subset N$$
as follows:

1. s_n is homotopic to the identity as a map to N.
2. $s_n(X) \subset X$ and $s_n(N - X) \subset (N - x)$.
3. s_n restricted to A is homotopic to w_n rel ∂A in $\hat{X}$.

These conditions define s_n up to homotopy in $\hat{M}$. We call any such map a *shuffle immersion.*

Lemma 3.3. *The map s_n satisfies the following properties:*

1. *The cover, M', of $\hat{M}$ associated to $(s_n)_*(\pi_1(N))$ is a homeomorphic to M and the lift, $s'_n : N \longrightarrow M'$, of s_n is homotopic to an embedding.*
2. *If $n \neq m$ then s_n and s_m are not homotopic in $\hat{M}$.*
3. *For each integer k, there is a homeomorphism*
$$h^n_k : \hat{M}(1,k) \longrightarrow M \supset \hat{M}$$
such that $h^n_k \circ d_{1,k} \circ s_n$ and s_m are homotopic in M. Furthermore the image of N under $h^n_k \circ d_{1,k} \circ s_n$ is contained in $\hat{M}$ and $h^n_k \circ d_{1,k} \circ s_n$ and s_n are homotopic in $\hat{M}$.

Proof.

1. This is essentially Proposition 9.1 of [1]. See Remark 2 below.

2. Given a curve, γ, and surface, S, in N let $\#(\gamma, S)$ be number of times γ intersects S and let $i(\gamma, S)$ be the minimum of $\#(\gamma', S)$ where γ' ranges over all curves homotopic to γ.

   Now choose a γ in N such that $i(\gamma, \partial_1 X) = k > 0$. Let A_0 be the annulus $\{0\} \times [0,1] \times S^1$ in $X \subset N$. We show that $i(s_n(\gamma), A_0) = nk$. First it is clear that $i(s_n(\gamma), A_0) \leq nk$ so we only need to show that $i(s_n(\gamma), A_0) \geq nk$. Let $\partial_1' X$ be the unique component of the pre-image of $\partial_1 X$ in M' that intersects the image of s_n'. Then $i(s_n'(\gamma), \partial_1' X) = k$. Let $\tilde{A}_0$ be the pre-image of A_0 in N under the immersion s_n and $\tilde{A}_0'$ the pre-image of A_0 in the cover M'. The map s_n can be chosen such that $\tilde{A}_0$ has exactly n components which are mapped to n distinct components of $\tilde{A}_0'$. Each of these n components will be parallel to $\partial_1' X$ and therefore each component will have k intersections with γ. This implies that $i(s_n'(\gamma), \tilde{A}_0') = i(s_n(\gamma), A_0) \geq nk$, as desired. The intersection number is a homotopy invariant so s_n cannot be homotopic to s_m if $n \neq m$.

3. On $\hat{X}(1,k) \subset \hat{M}(1,k)$ we let $h_k^n = h_{k(n+1),kn}$. On the remainder of $\hat{M}(1,k)$ we let $h_k^n = d_{1,k}^{-1}$. The statements then follow from Lemma 3.2. q.e.d.

Remark 1. The main point of this lemma, which can be lost in all the notation, is to compare the homotopy classes of the maps s_n and s_m after Dehn filling. The difficulty is that while the Dehn filled manifolds, $\hat{M}(1,k)$, are all homeomorphic to M, all the homeomorphisms from $\hat{M}(1,k)$ to M are not homotopic. The maps, h_k^n, pin down the homotopy class.

Remark 2. Our shuffle immersion is very similar to the *primitive shuffle* defined in [1]. They are both homotopy equivalences that are homotopic to embeddings outside a solid torus (or a collection of solid tori). There are two differences that are significant here. First, in a primitive shuffle the image of the map in the solid torus is not required to avoid the core curve. Rather, the first part of Proposition 9.1 in [1] is to show that a primitive shuffle is homotopic to a map that has the properties of a shuffle immersion. The reason for this difference in the two definitions is that we need to keep track of the homotopy class of the map in $\hat{M}$ while in [1] this homotopy class is not important. The second difference is that for a primitive shuffle all of the "shuffling" takes place in tori contained in the characteristic submanifold; i.e. the solid torus intersects the boundary, ∂N, in at least 3 annuli which are homotopically distinct in ∂N. Here the difference in definitions is because all essential, boundary incompressible annuli do not lead to the bumping of distinct component of $MP(\pi_1(N))$, which is studied in [1], but do lead to the self-bumping phenomena investigated here. Allowing this broader class of tori does not affect the proof of Proposition 9.1 in [1].

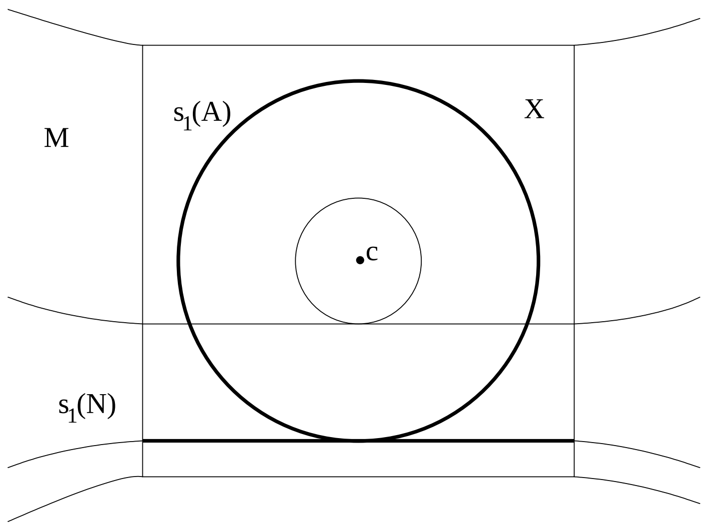

Figure 4: The map s_1 immerses N in M and is not homotopic to an embedding in $\hat{M}$.

4. Self-bumping

We now use the topology we developed in §3. With the same assumptions as in §3 we fix a shuffle immersion, $f = s_d$, with $d > 0$. Both M and $\hat{M}$ satisfy the conditions of Thurston's hyperbolization theorem (see Lemma 2.5.10 in [8]) and we fix a complete, minimally parabolic hyperbolic structure, $\hat{M}_\infty$, on $\hat{M}$ with holonomy representation $\hat{\rho}_\infty$. We also let N_∞ be the complete hyperbolic structure with boundary on N obtained as the pull-back by f of the metric $\hat{M}_\infty$ on $\hat{M}$.

We now set up a notational system that will hold for the remainder of the paper. Note that while M is the interior of N, we will continuously be examining immersions of N in M. On M we will study complete infinite volume, hyperbolic metrics while on N the hyperbolic metrics will define a compact Riemannian manifold with boundary. For an index α, N_α is a hyperbolic structure on N and ρ_α will be the associated holonomy representation. The hyperbolic structure N_α is not uniquely determined by ρ_α; however, in practice we will choose a single such structure. As we noted in the introduction, if $\rho_\alpha \in AH(\pi_1(N))$ then M_α is a complete hyperbolic structure, marked by N. As N_α has the same holonomy as M_α there will be an isometric immersion, $f_\alpha : N_\alpha \longrightarrow M_\alpha$, with f_α a homotopy equivalence. In other words, f_α is a marking map. Let c_α denote the geodesic representative of c in M_α.

Lemma 4.1. *There exists a neighborhood U of N_∞ in $\mathcal{H}(N)$ such that, if $N_\alpha \in U$ and the associated holonomy ρ_α is also in $MP(\pi_1(N))$, then $f_\alpha(N_\alpha) \cap c_\alpha = \emptyset$.*

Proof. Let γ be a non-trivial closed curve in N that is not commensurable to c. By compactness, the hyperbolic structure N_∞ has finite diameter. Therefore there exists a K such that for every $p \in N$ there exists a closed curve γ_p, freely homotopic to γ, with $p \in \gamma_p$ and the length of γ_p less than K in N_∞. We choose U small enough such that all structures in the neighborhood are 2-biLipschitz from N_∞. The Margulis lemma implies that there exists an ϵ such that, for any complete hyperbolic 3-manifold, if a homotopically non-trivial closed curve intersects a homotopically distinct geodesic of length $< \epsilon$ it has length $> 3K$. Furthermore, since the length of curves is continuous on $R(\pi_1(N))$, we can further shrink U so that the curve c_α has length $< \epsilon$ and therefore $f_\alpha(\gamma_p)$, which has length $< 2K$, does not intersect c_α; implying that $p \notin c_\alpha$. q.e.d.

Let B be the component of $MP(\pi_1(N))$ such that every $\rho_\alpha \in B$ has a marking, f_α, which is an embedding.

Lemma 4.2. *For the shuffle immersion f, there exists a sequence of hyperbolic structures N_k with holonomy representations ρ_k, such that:*

1. *$N_k \to N_\infty$ and $\rho_k \to \rho_\infty$.*
2. *There exist homeomorphisms $h_k : M_k \longrightarrow M \supset \hat{M}$ such that $h_k(c_k) = c$ and f and $h_k \circ f_k$ are homotopic in $\hat{M}$.*
3. *$\rho_k \in B$.*

Proof.

1. For large n, let $M_n = \hat{M}_\infty(1,n)$ be the manifolds obtained by performing hyperbolic Dehn surgery on $\hat{M}_\infty$ as in Theorem 2.2. Since $f_\infty(N_\infty)$ is contained in a compact subset of $\hat{M}$, Theorem 2.2 also implies that the maps $d_{1,n} : \hat{M}_\infty \longrightarrow M_n$ restricted to $f_\infty(N_\infty)$ are K_n-biLipschitz with $K_n \to 1$ as $n \to \infty$. Therefore the hyperbolic structures, N_n, defined by pulling back the hyperbolic metric on M_n by $d_{1,n} \circ f_\infty$ converge to N_∞. Finally, if $N_k \to N_\infty$ then $\rho_k \to \rho_\infty$.

2. These homeomorphisms are supplied by Lemma 3.3. The fact that M_k is a *hyperbolic* Dehn-filling allows us to choose the h_k such the geodesic c_k is mapped to c.

3. Recall that f is homotopic to an embedding in M and therefore $h_k \circ f_k$ is also homotopic to an embedding. Since h_k is a homeomorphism, f_k is also homotopic to an embedding and therefore $\rho_k \in B$. q.e.d.

The following lemma will be used to detect when two representations are not contained in the same component of $V \cap B$.

Lemma 4.3. *Let U be a neighborhood of N_∞ that satisfies the conclusion of Theorem* 2.1 *and Lemma* 4.1 *and let V be the image of U under the holonomy map. Let N_0 and N_1 be hyperbolic structures in U with holonomy ρ_0 and ρ_1, both in $V \cap MP(\pi_1(N))$. Also assume that $h_i : M_i \longrightarrow M$, $i = 0,1$, are homeomorphisms that are homotopy inverses of $f_i|_M$ and $h_i(c_i) = c$. If ρ_0 and ρ_1 are in the same path component of $V \cap MP(\pi_1(M))$ then $h_0 \circ f_0$ and $h_1 \circ f_1$ are homotopic in $\hat{M}$.*

Proof. Choose a smooth path, ρ_t, $0 \le t \le 1$, connecting ρ_0 and ρ_1 in $V \cap MP(\pi_1(M))$. The ρ_t are all in the same component of $MP(\pi_1(M))$ so the ρ_t are all quasiconformally conjugate to ρ_0. Indeed, there is a continuous family ϕ_t of quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$, so that ϕ_t conjugates ρ_0 to ρ_t, and $\phi_1 = h_1^{-1} \circ h_0$. By [13], Theorem B.21, there is a smooth family of biLipschitz homeomorphisms $\Phi_t : M_0 \longrightarrow M_t$. Now set $h_t = h_0 \circ \Phi_t^{-1}$.

The push-forward of the hyperbolic metrics on M_t to M is a smoothly changing family of metrics on M so the geodesic representative of c will change continuously. Furthermore, as all the c_t are short geodesics, they will be simple. Hence $h_t(c_t)$ is an isotopy of c in M. We can therefore modify the h_t such that $h_t(c_t) = c$. Then $h_t \circ f_t$ will vary continuously in t.

By Theorem 2.1 we have a path of structures N_t in U with holonomy ρ_t. By Lemma 4.1 $f_t(N_t) \cap c_t = \emptyset$ so $h_t \circ f_t$ is a homotopy between $h_0 \circ f_0$ and $h_1 \circ f_1$ in $\hat{M}$. q.e.d.

We next apply Lemma 4.3 to show that distinct shuffle immersions force $V \cap B$ to be disconnected.

Lemma 4.4. *Let $f, f' : N \longrightarrow \hat{M} \subset M$, be distinct shuffle immersions. Assume that there exist minimally parabolic structures $\hat{M}_\infty$ and $\hat{M}'_\infty$ on $\hat{M}$ such that the pulled-back hyperbolic structures N_∞ and N'_∞ are isometric and hence define the same holonomy representation, ρ_∞. Then for every small neighborhood V of ρ_∞, $V \cap B$ is disconnected.*

Proof. Let M_n, N_n, f_n, h_n, and ρ_n and M'_n, N'_n, f'_n, h'_n, and ρ'_n be the hyperbolic structures, isometric immersions and holonomy representations given by Lemma 4.2 for f and f', respectively. Choose an open neighborhood V of ρ_∞ given by Lemma 4.1.

There exists integers n and m such that $\rho_n, \rho'_m \in V$. The intersection $V \cap B$ is an open subset of the manifold B so the connected components of $V \cap B$ are path connected. If ρ_n and ρ'_m are in the same component of $V \cap B$ then Lemma 4.3 implies that $h_n \circ f_n$ and $h'_m \circ f'_m$ are homotopic in $\hat{M}$. On the other hand, by Lemma 4.2, $h_n \circ f_n$ and $h'_m \circ f'_m$ are homotopic in $\hat{M}$ to f and f', respectively. Since, f and f' aren't homotopic in $\hat{M}$ we have a contradiction. q.e.d.

We now prove our main theorem.

Theorem 4.5. *Let N be a compact, orientable, atoroidal, irreducible 3-manifold with boundary. Suppose that N contains an essential, boundary incompressible annulus whose core curve is not homotopic into a torus boundary component of ∂N. Let B be a component of the interior of $AH(\pi_1(N))$. Then there is a representation ρ in $\overline{B}$ such that for any sufficiently small neighborhood V of ρ in $AH(\pi_1(N))$ the set $V \cap B$ is disconnected.*

Proof. We recall our standing assumption that if $\rho \in B$ then the marking map, $f_\rho : N \longrightarrow M_\rho$, has a homotopy inverse that is a homeomorphism onto the interior of N. If we want to show self-bumping at a different component, B', we find a new manifold N', homotopy equivalent to N, such that N' and B' have the above property. With the exception of N being irreducible, all the topological assumptions we have made only depend on the homotopy type of N. Since a hyperbolic manifold is automatically irreducible, N' will also be atoroidal, irreducible and contain and essential, boundary incompressible annulus. In particular if one component of $MP(\pi_1(N)$ self-bumps then every component of $MP(\pi_1(N))$ will self-bump.

By Lemma 3.3, there is a non-trivial shuffle immersion $f : N \longrightarrow \hat{M} \subset M$ and f lifts to an embedding f' in the cover, M', associated to $f_*(\pi_1(N))$ with M' homeomorphic to M. Let $\hat{M}_\infty$ be a minimally parabolic structure on $\hat{M}$ which defines a hyperbolic structure M'_∞ on $M' = M$. We use f to pull back a hyperbolic structure, N_∞, on N. Then $f_\infty : N_\infty \longrightarrow \hat{M}_\infty$ is an isometric immersion and $f'_\infty : N_\infty \longrightarrow M'_\infty$ is an isometric embedding. Let ρ_∞ denote the holonomy of M'_∞. To finish the proof we construct a hyperbolic structure $\hat{M}'_\infty$ on $\hat{M}$ such that M'_∞ covers $\hat{M}'_\infty$ and f'_∞ descends to an isometric embedding $f' : N_\infty \longrightarrow \hat{M}'_\infty$.

Let δ denote the parabolic isometry $\rho_\infty(c)$. Because M'_∞ is geometrically finite and the image of f'_∞ is compact, in $\mathbb{H}^3$ we can find two disjoint totally geodesic halfspaces, H_1 and H_2, with the following properties.

- $\overline{H}_1 \cap \overline{H}_2$ is the fixed point of δ;
- $(H_1 \cup H_2)/ < \delta >$ embeds in M'_∞ under the covering map;
- the set $(H_1 \cup H_2)/\rho_\infty(\pi_1(N))$ is disjoint from the image of f'_∞.

Let γ be a parabolic commuting with δ and so that δ is a homeomorphism between H_1 and the complement of H_2. Then by the second Klein-Maskit combination theorem (see [11]) the group generated by $\rho_\infty(\pi_1(N))$ and γ is discrete, torsion free, geometrically finite and uniformizes $\hat{M}$ (indeed, the manifold obtained is isometric to the result of removing $(H_1 \cup H_2)/\rho_\infty(\pi_1(N))$ and identifying the resulting boundary annuli by γ). Moreover, f'_∞ descends to an embedding in this new manifold, which we will denote by $\hat{M}'_\infty$.

Therefore f and f' satisfy the conditions of Lemma 4.4 which implies the theorem. q.e.d.

Corollary 4.6. *$\overline{B}$ is not a manifold.*

Proof. If $\overline{B}$ is a manifold then Theorem 4.5 implies that ρ_∞ is in the interior of $\overline{B}$, since it cannot be in the boundary. However, in [14], Sullivan proves that the interior of $\overline{B}$ is B. Since ρ_∞ is not in B, $\overline{B}$ is not a manifold. q.e.d.

In Theorem 4.5 we characterized when the components of $MP(\pi_1(N))$ self-bump. To do so we constructed a representation where this self-bumping occurs. In our next theorem we describe a sufficient condition for a representation to be a point of self-bumping. To describe it we will assume some knowledge of Kleinian groups.

We now allow N to contain more than one copy of X. In particular, assume that there are m pairwise disjoint embeddings of $(X, \partial X)$ in $(N, \partial N)$, labeled $X_1, \ldots, X_m$. As before we assume that each $\partial_1 X_i$ is an essential, boundary incompressible annulus and that each core curve, c_i is primitive and not homotopic to a boundary torus. We further assume that the c_i are homotopically distinct. For each i, $1 \leq i \leq m$, choose an integer, $n_i \geq 0$. There is then a shuffle immersion, $s_{n_1,\ldots,n_m}$, that wraps N around c_i, n_i times. Let $\mathcal{C}$ denote the collection $\{c_1, \ldots, c_m\}$.

Let $\hat{M} = M - \mathcal{C}$. If $\hat{\rho}$ is a minimally parabolic, geometrically finite uniformization of $\hat{M}$ then the space of all minimally parabolic hyperbolic structures on $\hat{M}$, with the same marking, is $QD(\hat{\rho})$, the *quasiconformal deformation space* of $\hat{\rho}$. The image of $(s_{n_1,\ldots,n_m})_*(\pi_1(N))$ in $\pi_1(\hat{M})$ defines a Kleinian subgroup Γ of $\hat{\Gamma} = \hat{\rho}(\pi_1(\hat{M}))$ that uniformizes M, and a representation $\rho = \hat{\rho} \circ (s_{n_1,\ldots,n_m})_*$, with image Γ. If $\hat{\rho}'$ is another representation in $QD(\hat{\rho})$ then $\hat{\rho}' \circ (s_{n_1,\ldots,n_m})_*$ is in $QD(\rho)$, the quasiconformal deformation space of ρ. Therefore $(s_{n_1,\ldots,n_m})_*$ defines a map between $QD(\hat{\rho})$ and $QD(\rho)$. Our previous work shows the following:

Theorem 4.7. *All representations in $QD(\rho)$ in the image of $QD(\hat{\rho})$ under $(s_{n_1,\ldots,n_m})_*$ are points of self-bumping for B if $n_i \neq 0$ for some i.*

Note that ρ will not be minimally parabolic, for the c_i will all be parabolic in $\Gamma = \rho(\pi_1(N))$. Let $c_i' = \{0\} \times \{1\} \times S^1 \subset \partial_0 X_i$. The quotient of the domain of discontinuity for Γ will be a conformal structure on $\partial N - \coprod c_i'$. As the pinched curves in ∂N are determined by the embeddings of the X_i, if $s_{n_1',\ldots,n_m'}$ is another shuffle immersion then the image of $(s_{n_1',\ldots,n_m'})_*$ will be the same quasiconformal deformation space, $QD(\rho)$. (While these maps have the same range, $(s_{n_1,\ldots,n_m})_*(\hat{\Gamma}) \neq (s_{n_1',\ldots,n_m'})_*(\hat{\Gamma})$.) On the other hand, each X_i has an involution which swaps the two components of $\partial_0 X_i$. By performing this involution on some (possibly all) of the X_i we get a new family of shuffle immersions. The bumping representations associated to these shuffle immersions will then lie in a different quasi-conformal deformation space.

We also remark that even in the case where N is an I-bundle, Theorem 4.7 is stronger than McMullen's result in [12]. In McMullen's theorem, all the c_i' must lie in the same component of ∂N. Here we have no such restriction.

We close with the following conjecture.

Conjecture 4.8. *A representation ρ is a point of self-bumping for B if and only if there is a non-empty collection of curves $\mathcal{C}$ (as above) in M, a non-trivial shuffle immersion s with respect to $\mathcal{C}$, and a uniformization $\hat{\rho}$ of $\hat{M} = M - \mathcal{C}$ so that $\rho = \hat{\rho} \circ s_*$.*

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