Math 6220
Homework 1
January 26 2007

Problem 1.1.4.3

Given: $|a| = 1$ or $|b| = 1$
By the definition of the absolute value of complex numbers,
we observe that $|\bar{a}| = |a| = a\bar{a} = 1$, $|\bar{b}| = |b| = b\bar{b} = 1$,

$$\Rightarrow |\tfrac{a-b}{1-ab}| = |\tfrac{a-b}{a\bar{a}-ab}| = |\tfrac{a-b}{\bar{a}(a-b)}| = \tfrac{1}{|\bar{a}|}|\tfrac{a-b}{a-b}| = 1$$

$$\Rightarrow |\tfrac{a-b}{1-\bar{a}b}| = |\tfrac{a-b}{b\bar{b}-\bar{a}b}| = \tfrac{1}{|b|}|\tfrac{a-b}{\bar{b}-\bar{a}}| = |\tfrac{a-b}{\bar{b}-\bar{a}}| = 1$$

$$\textbf{since } |\tfrac{a-b}{\bar{a}-\bar{b}}|^2 = \tfrac{(a-b)(\bar{a}-\bar{b})}{(\bar{a}-\bar{b})(\bar{\bar{a}}-\bar{\bar{b}})} = \tfrac{(a-b)(\bar{a}-\bar{b})}{(\bar{a}-\bar{b})(a-b)} = 1$$

Problem 1.1.5.1

Prove that $|\frac{a-b}{1-\bar{a}b}| < 1$ if $|a| < 1$ and $|b| < 1$.

Solution :

$$\begin{aligned}
a\bar{a}(1 - b\bar{b}) &< 1 - b\bar{b} \\
a\bar{a} - a\bar{a}b\bar{b} &< 1 - b\bar{b} \\
a\bar{a} + b\bar{b} &< 1 + a\bar{a}b\bar{b} \\
a\bar{a} - a\bar{b} - \bar{a}b + b\bar{b} &< 1 - a\bar{b} - \bar{a}b + a\bar{a}b\bar{b} \\
(a-b)(\bar{a}-\bar{b}) &< (1-\bar{a}b)(1-a\bar{b}) \\
|a-b|^2 &< |1-\bar{a}b|^2
\end{aligned}$$

$\square$

Problem 1.2.1.2

Prove that the points a_1, a_2, a_3 are the vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$.
Proof:

(⇒) To make our lives easier, we shift the center of our arbitrary equilateral triangle $\triangle[a_1a_2a_3]$ to the origin. We do this by letting $z_0 = \frac{a_1+a_2+a_3}{3}$ be our center of mass and subtracting this point from each vertex. Notice that the distance from z_0 to each a_i is a constant length r and the angle between each point is $\frac{2\pi}{3}$. That is,

$$
\begin{aligned}
a_1 - z_0 &= re^{i\theta} \\
a_2 - z_0 &= re^{i(\theta+2\pi/3)} \\
a_2 - z_0 &= re^{i(\theta-2\pi/3)}
\end{aligned}
$$

It is easy to see that $(a_1 - z_0)^2 = (a_2 - z_0)(a_3 - z_0)$, and by plugging in $z_0 = \frac{a_1+a_2+a_3}{3}$, we get the desired $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$.

(⇐) We can begin by simplifying the problem via shifting the vertex a_1 to the origin. The triangle in question is now $\triangle[(0)(a_2 - a_1)(a_3 - a_1)]$. Then

$$
\begin{aligned}
a_1^2 + a_2^2 + a_3^2 - a_1a_2 - a_2a_3 - a_3a_1 = 0 &\Leftrightarrow (a_2 - a_1)^2 + (a_3 - a_1)^2 = (a_2 - a_1)(a_3 - a_1) \\
&\Leftrightarrow \frac{a_2 - a_1}{a_3 - a_1} + \frac{a_3 - a_1}{a_2 - a_1} = 1 \\
&\Leftrightarrow x + \frac{1}{x} = 1 \\
&\Leftrightarrow x^2 - x + 1 = 0 \\
&\Leftrightarrow \frac{a_2 - a_1}{a_3 - a_1} = x = e^{\pm i\pi/3}.
\end{aligned}
$$

This is sufficient to show $\triangle[(0)(a_2 - a_1)(a_3 - a_1)]$, and thus $\triangle[a_1a_2a_3]$ is an equilateral triangle.

Problem 1.2.2.4

Since h is not a multiple of n, $\omega^h \neq 1$.
Using the formula of the geometric series, we have

$$1 + \omega^h + \cdots + \omega^{h(n-1)} = \frac{1 - \omega^{hn}}{1 - \omega^h}.$$

Hence, $1 + \omega^h + \cdots + \omega^{h(n-1)} = 0$ if and only if $\omega^{hn} = 1$ and this is true since $h \in \mathbb{Z}$. Indeed,

$$\omega^{hn} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{hn} = \cos 2\pi h + i \sin 2\pi h = 1.$$