Notes and problems on infinite sets and countability

A set X is *infinite* if there exists a map from X to X that is injective but not surjective.

Theorem 1 *If X is infinite there is an injective map from $\mathbb{N}$ to X.*

Proof. Let $\phi : X \to X$ be injective but not surjective. We inductively define an injective map $\psi : \mathbb{N} \to X$ as follows. Define $\psi(1)$ to be an element of $X \backslash \phi(X)$. Now assume ψ has been defined on $\{1, \ldots, n\}$ and that $\psi(k) \in \phi^{k-1}(X) \backslash \phi^k(X)$ for $k \in \{1, \ldots, n\}$. Now define $\psi(n+1)$ to be an element of $\phi^k(X) \backslash \phi^{k+1}(X)$.

This defines ψ on all of $\mathbb{N}$. The map is injective since

$$(\phi^n(X) \backslash \phi^{n+1}(X)) \cap (\phi^m(X) \backslash \phi^{m+1}(X)) = \emptyset$$

if $n \neq m$. $\boxed{1}$

A set X is *countable* if there exists a bijection from $\mathbb{N}$ to X.

Problem 1 Show that:

- $\mathbb{Z}$ is countable.
- The union of two countable sets is countable.

Theorem 2 *The product of two countable sets is countable.*

Proof. We just need to show that $\mathbb{N} \times \mathbb{N}$ is countable. We can write $\mathbb{N} \times \mathbb{N}$ in a list:

$$(1,1), (1,2), (2,1), (1,3), (2,2), (3,1), (4,1), (2,3), (3,2), (4,1), \ldots$$

$\boxed{2}$

Problem 2 Explicitly write down a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$.

Theorem 3 *An infinite subset of a countable set is countable.*

Proof. We can assume that the countable set is $\mathbb{N}$. Let A be an infinite subset of $\mathbb{N}$. Every subset of $\mathbb{N}$ has a least element. We use this fact to inductively define a bijection $\psi : \mathbb{N} \to A$.

Define $\psi(1)$ to be the least element of A and let $A_1 = A \backslash \{\psi(1)\}$. Now assume we have defined $\psi(k)$ and A_k for $k \in \{1, \ldots, n\}$. Then we inductively define $\psi(n+1)$ to be the least element of A_n and define $A_{n+1} = A_n \backslash \{\psi(n)\}$. This define an injective map ψ from $\mathbb{N}$ to A.

We need to show that ψ is surjective. We claim that $\psi(n) \geq n$. We again use induction. Clearly $\psi(1) \geq 1$ since 1 is the least element of $\mathbb{N}$ and $\psi(1) \in A \subseteq \mathbb{N}$. Now assuming that $\psi(n) \geq n$ we will show that $\psi(n+1) \geq n+1$. Note that $\psi(n)$ is strictly less than any element of A_n so $\psi(n) < \psi(n+1)$ or $\psi(n) + 1 \leq \psi(n+1)$. Since $\psi(n) \geq n$ we have $\psi(n+1) \geq n+1$ as desired.

Since $\psi(n) \geq n$ for all $n \in \mathbb{N}$ we have $n \notin A_m$ for $n \leq m$. If $n \in A$ and $n \notin A_n$ then we must have $\psi(m) = n$ for some $m < n$ proving that ψ is surjective. $\boxed{3}$

Theorem 4 *Let $\mathcal{S}(X)$ be the set of all subsets of a set X. Then there is an injective map from X to $\mathcal{S}(X)$ but there is no surjective map from X to $\mathcal{S}(X)$. In particular there are infinite sets that are not countable.*

Proof. The map $x \mapsto \{x\}$ is an injective map from X to $\mathcal{S}(X)$.

Now we see there is no surjective map. Let $\psi : X \to \mathcal{S}(X)$ be a map and define a susbet A by

$$A = \{x | x \notin \psi(x)\}.$$

We claim that A is not in the image of ψ.

We work by contradiction and suppose there is an $x \in X$ such that $\psi(x) = A$. There are two cases.

Case 1: Suppose x is in A. Then $x \in \psi(X) = A$ so $x \notin A$ which is a contradiction.

Case 2: Suppose x is not in A. Then $x \notin \psi(X) = A$ so $x \in A$ which is again a contradiction.

Therefore there does not exist an $x \in X$ with $\psi(x) = A$ and ψ is not surjective. $\boxed{4}$

We'd also like to prove that the real numbers are not countable. We first give a definition of a real numbers. Our definition is not the usual one but it is convenient for showing that $\mathbb{R}$ is not countable.

A *real number* is a function $f : \mathbb{Z} \to \{0, 1, \ldots 9\}$ with the following properties:

1. There exits an $N > 0$ such $f(n) = 0$ if $n > N$;

2. For every n such that $f(n) = 9$ there is an $m < n$ such that $f(m) \neq 9$.

Here is an example. There real number 32.71 is represented by the function f with $f(1) = 3$, $f(0) = 2$, $f(-1) = 7$, $f(-2) = 1$ and $f(n) = 0$ for $n \notin \{, 0, -1, -2\}$. A more complicated example is the number 1/7. This number is represented by a function f with $f(-1) = 1$, $f(-2) = 4$, $f(-3) = 2$, $f(-4) = 8$, $f(-5) = 5$, $f(-6) = 7$, $f(n) = f(n+6)$ if $n < -6$ and $f(n) = 0$ if $n \geq 0$.

Theorem 5 $\mathbb{R}$ *is uncountable.*

Proof. Let ϕ be a map from $\mathbb{N}$ to $\mathbb{R}$ and let $f_n = \phi(n)$. We will show that ϕ is not surjective. Define $g \in \mathbb{R}$ by setting $g(n)$ to be some element of $\{0, 1, \ldots, 8\} \backslash \{f_n(n)\}$ if $n < 0$ and $g(n) = 0$ if $n \geq 0$. Then $g \neq f_n$ for any $n \in \mathbb{N}$ since $g(n) \neq f_n(n)$. Therefore ϕ is not surjectve. $\boxed{5}$

The number $f \in \mathbb{R}$ *eventually periodic* if there exists and $N \in \mathbb{Z}$ and a $k \in \mathbb{N}$ such that $f(n) = f(n-k)$ if $n < N$. The *period* of f is k.

Problem 3 Show that f is rational if and only if f is eventually periodic. (**Hint:** To show that and eventually periodic f is rational show $10^k f - f$ is rational where k is the period of f. It is harder to show that a rational number has a eventually periodic decimal expansion is harder.)

If f and g are real numbers we define $f > g$ if there exists an $n_0 \in \mathbb{Z}$ such that $f(n) = g(n)$ for all $n > n_0$ and $f(n_0) > g(n_0)$.

Problem 4 Let f_0 and f_1 be real numbers. Show that there exists a rational number g_0 and an irrational number g_1 such that $f_0 < g_i < f_1$ for $i = 1, 2$.