Final Exam, Math 5210 Spring 2008
Due by 5 PM, April 29 in JWB 303

Write in complete sentences and justify your work. You may consult books and your notes but otherwise you should work independently and all answers should be in your own words. You should reference any books that you use. All 5 problems will be waited equally.

1. Let $E = [0,1] \cap \mathbb{Q}$. Find an open set U such that $E \subset U$ and $m(U) < 1/2$.

2. Let $f(x) = x^2$. Find a simple function $s(x)$ such that $0 \le s(x) \le f(x)$ and
$$\int_{[0,1]} f(x)dx - \int_{[0,1]} s(x)dx < 1/5.$$

3. Find sequence of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ such that $f_n(x) \to 0$ for all $x \in [0,1]$ but
$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx = 1.$$

4. Let $f(x) = x^2 - x$ and let $e_n(x) = e^{2\pi\imath nx}$ be the standard orthonormal basis for $L^2[0,1]$. Show that the Fourier coefficients $\hat{f}(n) = (f, e_n)$ are
$$f(x) = \begin{cases} -\frac{1}{6} & n = 0 \\ \frac{1}{2\pi^2 n^2} & n \neq 0 \end{cases}$$
and use this to show that
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

5. Let $f_n(x) = \sqrt{2}\sin \pi nx$ be a family of functions in $L^2[0,1]$. As we saw in class this is an orthornormal basis for $L^2[0,1]$. Let $g(x) = 1$ be a constant function. Calculate the Fourier coefficients
$$(g, f_n)$$
for this basis. For which values of n are the coefficients zero?

   Let
$$g_n(x) = \sum_{k=0}^{n} \frac{4}{\pi(2k+1)} \sin \pi(2k+1)x.$$
   Show that $g_n(x) \to g(x)$ in $L^2[0,1]$ but that $g_n(x)$ doesn't converge to $g(x)$ pointwise.