Extra problems for Homework 1.

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex valued function. Recall that we have defined

$$\frac{\partial f}{\partial z} = f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \imath \frac{\partial f}{\partial y}\right)$$

and

$$\frac{\partial f}{\partial \overline{z}} = f_{\overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \imath \frac{\partial f}{\partial y}\right).$$

Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be another complex valued function and define h to be the composition $h = f \circ g$.

1. Show that $h_z(w)$ and $h_{\overline{z}}(w)$ only depend on $g_z(w)$, $g_{\overline{z}}(w)$, $f_z(g(w))$ and $f_{\overline{z}}(g(w))$. (Hint: Why does a similar statement hold for the partial derivatives of h with respect to x and y?)

   Solution: Both $h_z(w)$ and $h_{\overline{z}}(w)$ are determined by $h_x(w)$ and $h_y(w)$. By the chain rule these are both functions of $g_x(w)$, $g_y(w)$, $f_x(g(w))$ and $f_y(g(w))$. Since $g_x(w) = g_z(w) + g_{\overline{z}}(w)$, $g_y(w) = \imath(g_z(w) - g_{\overline{z}}(w))$, $f_x(g(w)) = f_z(g(w)) + f_{\overline{z}}(g(w))$ and $f_y(g(w)) = \imath(f_z(g(w)) - f_{\overline{z}}(g(w)))$ we get that $h_z(w)$ and $h_{\overline{z}}(w)$ are function of $g_z(w)$, $g_{\overline{z}}(w)$, $f_z(g(w))$ and $f_{\overline{z}}(g(w))$.

2. Show that:

   (a) $h_z = (f_z \circ g)g_z + (f_{\overline{z}} \circ g)\overline{g_{\overline{z}}}$

   (b) $h_{\overline{z}} = (f_z \circ g)g_{\overline{z}} + (f_{\overline{z}} \circ g)\overline{g_z}$

   (Hint: Use (1) to reduce the calculation to the special functions $f(z) = az + b\overline{z}$ and $g(z) = cz + d\overline{z}$.)

   Solution: Let $\hat{f}(z) = f_z(g(w))z + f_{\overline{z}}(g(w))\overline{z}$ and $\hat{g}(z) = g_z(w)z + g_{\overline{z}}(w)\overline{z}$. Note that $f_z(g(w)) = \hat{f}_z(g(w))$, $f_{\overline{z}}(g(w)) = \hat{f}_{\overline{z}}(g(w))$, $g_z(w) = \hat{g}_z(w)$, $g_{\overline{z}}(w) = \hat{g}_{\overline{z}}(w)$. Therefore, by problem 1, if $\hat{h} = \hat{f} \circ \hat{g}$ then $h_z(w) = \hat{h}_z(w)$ and $\hat{h}_{\overline{z}}(w) = \hat{h}_{\overline{z}}(w)$.

   Calculating the composition we see that

   $$\begin{aligned} \hat{h}(w) &= f_z(g(w))(g_z(w)z + g_{\overline{z}}(w)\overline{z}) + f_{\overline{z}}(g(w))\overline{(g_z(w)z + g_{\overline{z}}(w)\overline{z})} \\ &= (f_z(g(w))g_z(w) + f_{\overline{z}}(g(w))\overline{g_{\overline{z}}(w))})z + (f_z(g(w))g_{\overline{z}}(w) + f_{\overline{z}}(g(w))\overline{g_z(w)}\overline{z} \end{aligned}$$

   and therefore

   $$h_z(w) = \hat{h}_z(w) = f_z(g(w))g_z(w) + f_{\overline{z}}(g(w))\overline{g_{\overline{z}}(w))}$$

   and

   $$h_{\overline{z}}(w) = \hat{h}_{\overline{z}}(w) = f_z(g(w))g_{\overline{z}}(w) + f_{\overline{z}}(g(w))\overline{g_z(w)}.$$