

Hyperbolic geometry in dimensions 2 and 3 Problems

1. Show that there doesn't exist a conformal metric on $\widehat{\mathbb{C}}$ such that $PSL_2\mathbb{C}$ acts isometrically.
2. (a) Find a subgroup Γ of $PSL_2\mathbb{C}$ that is isomorphic to $SO(3)$.
 (b) Show that Γ acts transitively on $\widehat{\mathbb{C}}$. (For all $z, w \in \widehat{\mathbb{C}}$ there exists a $\phi \in \Gamma$ with $\phi(z) = w$.)
 (c) Show that for any $z \in \widehat{\mathbb{C}}$ the subgroup of Γ that fixes z is isomorphic to $SO(2)$.
 (d) Find a conformal metric on $\widehat{\mathbb{C}}$ where Γ acts isometrically.
3. By the Riemann mapping theorem every simply connected proper subset $\omega \subset \mathbb{C}$ has a holomorphic diffeomorphism to Δ (or equivalently \mathbb{U}) so for an such region we can pull back the hyperbolic metric via the Riemann map to find a hyperbolic metric on Ω . Let $\mathbb{S} = \{z \in \mathbb{C} | 0 < \text{Im } z < \pi\}$. Find the Riemann map from \mathbb{S} to \mathbb{U} and use it to find the hyperbolic metric on \mathbb{S} .
4. Let $\Delta^\times = \{z \in \mathbb{C} | 0 < |z| < 1\}$ be the punctured disk. The universal cover (as a Riemann surface) of Δ^\times is \mathbb{U} (or Δ). Find the covering map. Find a conformal metric on Δ^\times whose pull back via the covering map is the hyperbolic metric.
5. Show the $(\mathbb{H}^2, d_{\mathbb{H}^2})$ is complete as a metric space.
6. Let $B: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ be the bilinear form given by $B(x, y) = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$. Let $SO(n, 1)$ be matrices T with $\det T = 1$ and $B(Tv, Tw) = B(v, w)$ for all $v, w \in \mathbb{R}^{n+1}$.
 (a) Show that $SO(n, 1)$ is a Lie group. Find its Lie algebra.
 (b) Let $X \subset \mathbb{R}^{n+1}$ be the component of $B^{-1}(-1)$ with $x_{n+1} > 0$. Show that this is a smooth submanifold of dimension n .
 (c) At each point of X the tangent space is canonically identified with a subspace of \mathbb{R}^{n+1} so B is defined on each tangent space. Show that B is positive definite on the tangent space and hence defines a Riemannian metric on X .
 (d) Show that $SO(n, 1)$ acts isometrically on X with this metric. In fact this is another model of hyperbolic space and $SO(n, 1)$ is $\text{Isom}^+(\mathbb{H}^n)$.
7. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. M is a smooth manifold and $\langle \cdot, \cdot \rangle$ is a smoothly varying inner product on each tangent space. Given a Riemannian manifold one can define several types of curvature. In dimension two all of these different

types of curvature coincide. An exercise that every graduate student should do is to calculate the curvature in the hyperbolic plane.

To do this we rapidly discuss (without proof) some basic notions in Riemannian geometry. We begin with the Riemannian connection. This is map

$$\nabla: \Lambda(M) \times \Lambda(M) \rightarrow \Lambda(M)$$

where $\Lambda(M)$ is the space of smooth vector fields on the Riemannian manifold M . It is usually written $\nabla_Y X$ and satisfies the following properties:

- (a) $\nabla_{fY+gZ} X = f\nabla_Y X + g\nabla_Z X$;
- (b) $\nabla_Y(fX + Z) = (\nabla_Y f)X + f\nabla_Y X + \nabla_Y Z$;
- (c) $\nabla_X Y - \nabla_Y X = [X, Y]$;
- (d) $\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle X, \nabla_X Z \rangle$

(Here $\nabla_X f$ is the directional derivative of the function f in the direction X .) It is a fundamental result that every Riemannian metric has a unique Riemannian connection and there is a formula for it in a local chart. The standard vector fields E_1, \dots, E_n in \mathbb{R}^n define vector fields on the charts and then $g_{ij} = \langle E_i, E_j \rangle$ are smooth functions on the chart. If we think of the g_{ij} as being the coordinates of a matrix we can let g^{ij} be the coordinates of the inverse matrix. The Riemannian connection is determined by its values on a basis. (This follows from (a) and (b)). Let Γ_{ij}^k be smooth functions on the chart such that $\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k$. The Γ_{ij}^k are *Christoffel symbols* and they are given by the formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m} \right) g^{mk}.$$

The *curvature operator* is a map from $\Lambda(M) \times \Lambda(M)$ to the space of bundle maps from TM to itself. It is defined by

$$R(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X, Y]}.$$

At each point $p \in M$, $R(X, Y)$ is a linear map from $T_p M$ to itself. If $Z \in \Lambda(M)$ is a third vector field then $R(X, Y)Z$ is a vector field. If we have a fourth vector field $W \in \Lambda(M)$ then $\langle R(X, Y)Z, W \rangle$ is a smooth function on M . In an abuse of notation we write $R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$. We can now define the *sectional curvature*. This is a function on two-dimensional planes in the tangent space at a point p . In particular if P is a plane in $T_p M$ then we choose vector fields X and Y such that X and Y are a basis for P at p . Two vectors span a

parallelogram. Let $A(X, Y)$ be the area of this parallelogram. Then the sectional curvature $\kappa(P)$ is the value of $R(X, Y, X, Y)/(A(X, Y))^2$ at p . A fundamental result is that $\kappa(P)$ only depends on P and not the choice of X and Y .

Of course, there are many things that have been left unproved. They are not especially difficult; the difficulty is making the right definitions. A standard reference is do Carmo's book on Riemannian geometry. Another more concise reference is Milnor's Morse Theory book. Assuming everything that has been written is true we can now make some calculations for the hyperbolic plane.

- (a) Calculate Γ_{ij}^k for the hyperbolic metric in both \mathbb{U} and Δ .
 - (b) Calculate the sectional curvature of the hyperbolic plane.
8. Let $\alpha: I \rightarrow M$ be a smooth path. We would like to define $\nabla_{\alpha'}\alpha'$. However, ∇ is only defined for vector fields defined on all of M (or at least an open subset of M) so to define $\nabla_{\alpha'}\alpha'$ we need to extend α' to an open neighborhood. However, then we need to check that if X and X' are different extensions of α' then $\nabla_X X$ and $\nabla_{X'} X'$ agree along the image of α .

To see that this holds we observe the following: The vector field $\nabla_X Y$ at $p \in M$ only depends on Y and the value of X at p . (This follows from property (a) of the Riemannian connection.) In particular if X and X' agree at p then so does $\nabla_X Y$ and $\nabla_{X'} Y$. If $\alpha: I \rightarrow M$ is a path through $p \in M$ with tangent vector equal to Y at p and Y' is another vector field that agrees with Y on the image of α then again we have that $\nabla_X Y$ and $\nabla_X Y'$ agree at p . It follows that any extension X of α' will always give the same values on the image of α and $\nabla_{\alpha'}\alpha'$.

A geodesic on a Riemannian manifold is a smooth curve α with $\nabla_{\alpha'}\alpha' = 0$. In general the size of $\nabla_{\alpha'}\alpha'$ measure the *geodesic curvature* of α .

- (a) Show that geodesics in the \mathbb{H}^2 (as defined in class and in the notes) are geodesics in the Riemannian sense.
- (b) Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be a vector field in \mathbb{U} . Show that the flow lines of this vector are constant speed paths (where the constant depends on the path). Calculate $\nabla_X X$.
- (c) Let $\alpha_\theta(t) = e^{t \sin \theta + i\theta}$ be a path in upper half space model. Calculate $\nabla_{\alpha'_\theta}\alpha'_\theta$ and $\|\nabla_{\alpha'_\theta}\alpha'_\theta\|$. This last quantity is the geodesic curvature. (Hint: Use the calculation of $\nabla_X X$.)
- (d) Show that the path α_θ is a path of points of distance R from the imaginary axis where R is a function of θ . Find this function and show that the geodesic curvature of α_θ is $\tanh R$.

- (e) Show that the curvature of a circle of radius R is $\coth R$. (Hint: Work in the disk model and apply the above method to the vector field $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.)
- (f) Show that the curvature of a horocycle is 1. (Look at the vector field $\frac{\partial}{\partial x}$ in the upper half space model.)

9. For a function $g: \Omega \rightarrow \mathbb{R}$ the usual Laplacian is given by

$$\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4 \frac{\partial^2 g}{\partial z \partial \bar{z}}.$$

The function g is *harmonic* if $\Delta g = 0$.

- (a) Let $f: \Omega_0 \rightarrow \Omega_1$ be a conformal diffeomorphism. Show that $g: \Omega_1 \rightarrow \mathbb{R}$ is harmonic if and only if $g \circ f$ is harmonic.
- (b) The Laplacian with respect to a conformal metric ρ is given by $\Delta_\rho = \frac{1}{\rho^2} \Delta$. If $f: (\Omega_0, \rho_0) \rightarrow (\Omega_1, \rho_1)$ is an isometry and $g: \Omega_1 \rightarrow \mathbb{R}$ a smooth function show that $\Delta_{\rho_0}(g \circ f) = (\Delta_{\rho_1} g) \circ f$.
- (c) The curvature of a conformal metric is defined to be $\kappa_\rho(z) = -(\Delta_\rho \log \rho)(z)$. Show that $\kappa_{\rho_0}(z) = \kappa_{\rho_1}(f(z))$. In fact this metric is equal to the sectional curvature. (This can be shown via a direct calculation which is not pleasant. But perhaps there is a more clever way to do this.)
- (d) Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and define metric $\rho_\alpha(z) = \frac{\alpha}{2\pi} |z|^{\frac{\alpha}{2\pi} - 1}$. Calculate κ_{ρ_α} . What is the geometry of $(\mathbb{C}^\times, \rho_\alpha)$ near 0? (Hint: Let $\alpha = 2\pi n$ and find a metric such that the map $z \mapsto z^n$ is a local isometry.)

10. Given pairs (R_0, f_0) and (R_1, f_1) with $\ell_{\mathbf{P}}(R_0, f_0) = \ell_{\mathbf{P}}(R_1, f_1)$ and homeomorphisms $\phi_0, \phi_1: R_0 \rightarrow R_1$ such that

- both $\phi_0 \circ f_0$ and $\phi_1 \circ f_0$ are homotopic to f_1 ;
- both ϕ_0 and ϕ_1 restrict to an isometry from $V(\mathbf{P}, R_0) \rightarrow V(\mathbf{P}, R_1)$.

Show that there is a homotopy from ϕ_0 to ϕ_1 that is the identity on $V(\mathbf{P}, R_0)$.

11. Let $T: \mathbb{C} \rightarrow \mathbb{C}$ be an \mathbb{R} -linear map. Then $Tz = T_z z + T_{\bar{z}} \bar{z}$ where $T_z, T_{\bar{z}} \in \mathbb{C}$. Assume that $T_{\bar{z}} \neq 0$. Let $\mu = T_{\bar{z}}/T_z$. Let $\theta = \arg \mu/2$. Note that $\arg \mu$ defined mod 2π so θ is defined mod π . Show that

- (a) Show that T is invertible if and only if $|\mu| \neq 1$. Show that T is orientation preserving if $|\mu| < 1$ and orientation reversing if $|\mu| > 1$.

- (b) Show that $|Tz|/|z| \leq |T(e^{i\theta})|$ with equality if and only if $\arg z = \theta$. (This doesn't quite make sense since $\arg z$ is defined mod 2π and θ is defined mod π . What is the correct way to interpret this?)
- (c) Show that $|Tz|/|z| \geq |T(e^{i(\theta+\pi/2)})|$ with inequality if and only if $\arg z = \theta + \pi/2$. (Again, how should this be interpreted?)
- (d) Show that the dilation of T is $\frac{1+|\mu|}{1-|\mu|}$. This is the ratio of the maximal and minimal stretch.
12. Let Σ be a surface and $p \in \Sigma$ a point. Assume that there is a conformal structure \hat{X} on $\hat{\Sigma} = \Sigma \setminus \{p\}$ and a bounded holomorphic map $f: \hat{U} \rightarrow \mathbb{C}$ where U is a neighborhood of p in Σ and $\hat{U} = U \setminus \{p\}$. Show that there is a unique conformal structure X on Σ such that $X = \hat{X}$ on $\hat{\Sigma}$.
13. Here is another version of this fact. Let X be a Riemann surface and V a discrete collection of points on X . Let Y' be another Riemann surface and assume that $f: X \setminus V \rightarrow Y'$ is holomorphic. Then there exists a Riemann surface Y with $Y' \subset Y$, $Y \setminus Y'$ a discrete collection of points and such that f extends to a holomorphic map from X to Y .
14. Let Φ be a holomorphic quadratic differential on a Riemann surface X and assume that $z_0 \in X$ is a zero of Φ of order n . Show that there exists a chart (U, ψ) containing z_0 such that $\psi(z_0) = 0$ and in the chart Φ is represented by the function z^n .