Class exercises - Math 5520
January 31st, 2020

Groups

Let $G_0, G_1, \dots$ be a family of groups and assume for each $i < j$ there is homomorphism

$$\phi_{i,j} \colon G_i \to G_j$$

such that if $i < j < k$ then

$$\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}.$$

Let $\mathscr{G}$ be the disjoint union of the G_i. Define a relation on $\mathscr{G}$ as follows: If $g_i \in G_i$ and $g_j \in G_j$ then $g_i \sim g_j$ if there exists a k with $\phi_{ik}(g_i) = \phi_{jk}(g_j)$.

1. Show that $\sim$ is an equivalence relation.

For $g \in \mathscr{G}$ denote the equivalence class by $[g]$.

Define an operation on equivalence classes as follows: If $g_i \in G_i$ and $g_j \in G_j$ choose a k larger than i and j and set

$$[g_i] \cdot [g_j] = [\phi_{i,k}(g_i) \cdot \phi_{j,k}(g_j)].$$

2. Show that this is a well defined operation on the set of equivalence classes.

Let $\vec{G}$ be the set of equivalence classes.

3. Show that $\vec{G}$ with the given operation is a group.

4. Define maps

$$\phi_{i,\infty} \colon G_i \to \vec{G}$$

by $\phi_{i,\infty}(g) = [g]$. Show that the $\phi_{i,\infty}$ are homomorphisms and that $\phi_{i,\infty} = \phi_{j,\infty} \circ \phi_{i,j}$.

5. If all of the $\phi_{i,j}$ are the trivial homomorphism, show that $\vec{G}$ is the trivial group.

6. If all of the $\phi_{i,j}$ are isomorphisms, show that the G_i are isomorphic to $\vec{G}$.

7. Assume that G is another group and

$$\psi_i \colon G_i \to G$$

are injective homomorphisms with $\psi_i = \psi_j \circ \phi_{i,j}$ when $i < j$. Show that there exists an injective, homomorphism

$$\psi \colon \vec{G} \to G$$

with $\psi_i = \psi \circ \phi_{i,\infty}$. If for all $g \in G$ there exists an i and a $g_i \in G_i$ such that $\psi_i(g_i) = g$ then ψ is an isomorphism.

8. Assume that all of $G_i \cong \mathbb{Z}$. Find homomorphisms

$$\phi_{i,j}\colon G_i \to G_j$$

such that $\vec{G} \cong \mathbb{Q}$.

Topology

Let X be a topological space and

$$X_0 \subset X_1 \subset X_2 \subset \dots$$

a collection of open subspaces such that

$$X = \cup X_i.$$

The X_i are an *exhaustion* of X. Let

$$\iota_{i,j}\colon X_i \to X_j$$

be the inclusion maps. Fix a basepoint $x_0 \in X_0$. Let $G_i = \pi_1(X_i, x_0)$ and $\phi_{i,j} = (\iota_{i,j})_*$ be the induced homomorphisms.

9. Show that $\pi_1(X, x_0) \cong \vec{G}$.

Added on March 19th

We would like to construct a space X with $\pi_1(X, x_0) \cong \mathbb{Q}$. The construction will parallel the construction the group $\vec{G}$.

Let $X_0, X_1, X_2, \dots$ be (path-connected) topological spaces and for each $i < j$ let

$$\phi_{i,j}\colon X_i \to X_j$$

be an injective, continuous maps with $\phi_{i,j}(X_i)$ open in X_j such that if $i < j < k$ then

$$\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}.$$

Let $\mathscr{X}$ be the disjoint union of the X_i. We define a relation on $\mathscr{X}$ as we did for the collection of groups $\mathscr{G}$. If $x_i \in X_i$, $x_j \in X_j$ then $x_i \sim x_j$ if

- $i = j$ and $x_i = x_j$;
- $i < j$ and $\phi_{i,j}(x_i) = x_j$;
- $j < i$ and $\phi_{j,i}(x_j) = x_i$.

10. Show that $\sim$ is an equivalence relation.

Let $X = \mathscr{X} / \sim$ be the quotient space.

11. Show that the homeomorphisms

$$(\phi_{i,j})_* \colon \pi_1(X_i, x_i) \to \pi_1(X_j, x_j)$$

satisfy the conditions above on groups.

12. Let $\vec{G}$ be the group constructed there and show that $\pi_1(X, x_0)$ is isomorphic to $\vec{G}$. To simplify this problem (so that you can apply problem 7) you can assume that the $(\phi_{i,j})_*$ are injective.

13. Let

$$B^2 = \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\right\}$$

be the open disk and let

$$W = S^1 \times B^2.$$

Then W is a *solid torus*. As W and S^1 are homotopy equivalent $\pi_1(W, w_0) \cong \mathbb{Z}$. Show that for any integer $n \in \mathbb{Z}$ there is a injective, continuous map

$$\phi_n \colon W \to W$$

such that the homomorphism $(\phi_n)_*$ is multiplication by n.

14. Use the previous problem (and problems 8 and 9) to construct a space X with $\pi_1(X, x_0) \cong \mathbb{Q}$.

In problem 7 we assume that the homomorphisms $\phi_{i,j}$ are injective. While this condition is sufficient there is actually a weaker condition that is both necessary and sufficient. Namely we can assume that if $\psi_i(g_i) = id$ then there exists a $j > i$ such that $\phi_{i,j}(g_i) = id$ in G_j.

In the topological setup one can check that this condition always holds using that homotopies are compact. In particular, if X is the topological space constructed in problem 10 and $[f] \in \pi_1(X, [x_0])$ is trivial then not only is $[f] \in \pi_1(X_i, x_i)$ for some i but there is a $j > i$ such that $[f]$ is trivial in $\pi_1(X_j, x_j)$. This is exactly the algebraic condition of the previous paragraph.