Induced homomorphisms

Recall that if $h:(X,x_0)\to(Y,y_0)$ is continuous there is an induced homomorphism

$$h_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$$

$f:[0,1]\to X$

defined by $h_*([f])=[h\circ f]$.

$h\circ f:[0,1]\to Y$

How does this work when $X=Y=S^1$? Then $\pi_1(S^1,x_0)=\mathbb{Z}$ so for any map $h:S^1\to S^1$ the induced map is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$.

Let $\phi:\mathbb{Z}\to\mathbb{Z}$ be a homomorphism & assume that $\phi(1)=k$.

Then $\phi(0)=0$ so $0=\phi(0)=\phi(1+(-1))=\phi(1)+\phi(-1)=k+\phi(-1)\Rightarrow\phi(-1)=-k$.

If n is positive we have

$$\phi(n)=\phi(\underbrace{1+1+\dots+1}_{n\text{ times}})=\underbrace{\phi(1)+\phi(1)+\dots+\phi(1)}_{n\text{ times}}=n\cdot k$$

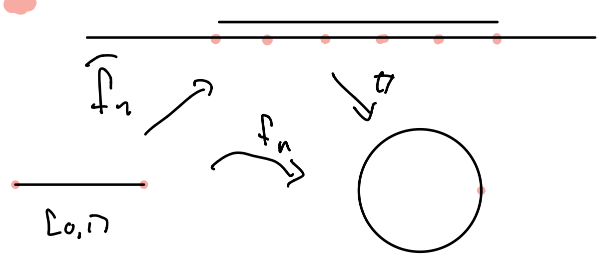
similarly $\phi(-n)=-n\cdot k$ so ϕ is uniquely determined by $\phi(1)$.

LEMMA Define $\phi_k:\mathbb{Z}\to\mathbb{Z}$ by $\phi_k(n)=k\cdot n$. If $\phi:\mathbb{Z}\to\mathbb{Z}$ is a homomorphism $\exists!\ k\in\mathbb{Z}$ st $\phi=\phi_k$.

Lets apply this to maps from S^1 to S^1. In homework we saw that every continuous map $g:S^1\to S^1$ is homotopic to a unique map $g_k:S^1\to S^1$ where $g_k([x])=[kx]$.

LEMMA $(g_k)_*=\phi_k$.

PROOF



$$[f_n]\in\pi_1(S_1,[0])$$

$$\begin{tikzcd}
 & S^1 \arrow[dr, "g_n"] & \\
{[0,1]} \arrow[ur, "f_1"] \arrow[rr, "f_n"] & & S^1
\end{tikzcd}$$

$$(g_n)_*([f_1]) = [g_n \circ f_1]$$
$$= [f_n]$$

In isomorphism $\pi_1(S^1, [0]) \cong \mathbb{Z}$
$[f_n] \longmapsto n$

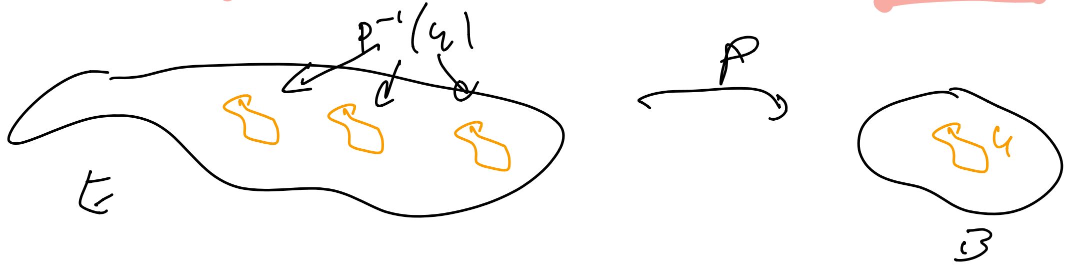
$\Rightarrow (g_n)_* = \phi_n$ ∎

$\boxed{\text{COR}}$ Let $g, h : (S^1, [0]) \to (S^1, [0])$.
Then $g \simeq h$ iff $g_* = h_*$.

$\boxed{\text{PF}}$ By lemma $g_{k_0} \simeq g_{k_1}$ iff $k_0 = k_1$
since $\phi_{k_0} = \phi_{k_1}$ iff $k_0 = k_1$.
But every map from $(S^1, [0]) \to (S^1, [0])$
is homotopic to a unique g_k.

COVERING SPACES

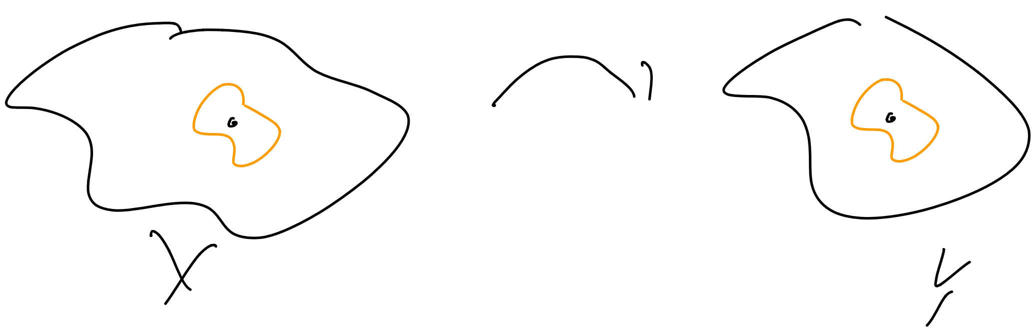
Let $p: E \to B$ be continuous and surjective. Then a nbd. U is **evenly covered** if $p^{-1}(U)$ is a disjoint union of open sets V_α such that restricted to each V_α, p is a homeomorphism to U.



p is a **covering map** if every point in B has an evenly covered nbd.
E is a **covering space**.

We have seen that the quotient map $\pi: \mathbb{R} \to S^1$ is a covering map.

A continuous map $f: X \to Y$ is a **local homeomorphism** if every $x \in X$ has a nbd V s.t. p restricted to V is a homeomorphism onto its image.



A covering map is a local homeomorphism: Given $x \in E$, let U be an evenly covered nbd of $p(x)$. Then $p^{-1}(U) = \amalg V_\alpha$ & x is contained in some V_α. By the definition of evenly covered p restricted to V_α is a homeomorphism onto its image.

The converse doesn't hold – a local homeo. need not be a covering map.

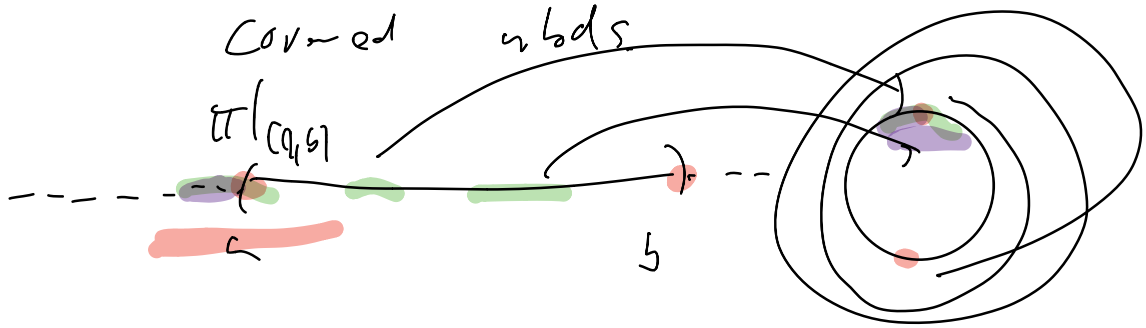
$\pi: \mathbb{R} \to S^1$ is a covering map

But

$\pi|_{(a,b)} : (a,b) \to S^1$ is a local homeo.

but ~~not~~ a covering map.

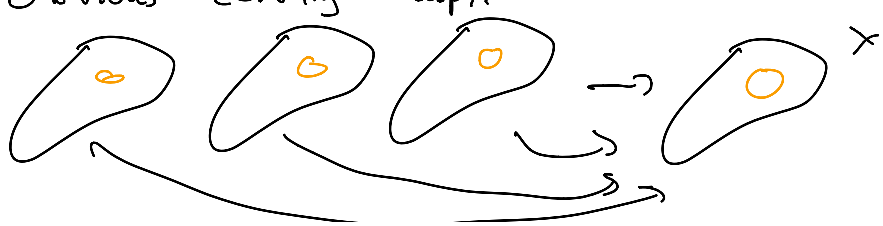
Since $\pi(a)$ & $\pi(b)$ don't have evenly covered nbds.



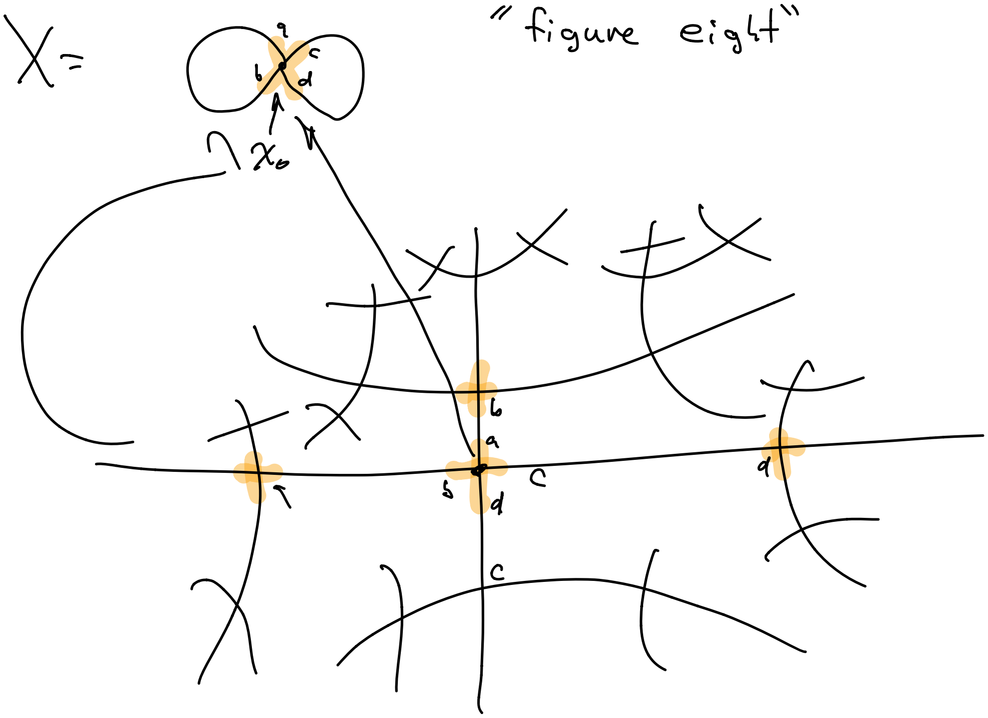
EXAMPLES The maps $g_n: S^1 \to S^1$ are covering maps.

Silly examples:

- The identity map is a covering map.
- A disjoint union of copies of X is a covering space of X (with obvious covering map).



$X =$



"figure eight"

$\tilde{X}$