LIFTING LEMMAS $p: E \to B$ a covering spaces.

Given $f: X \to B$ when can we find a lift $\tilde{f}: X \to E$?

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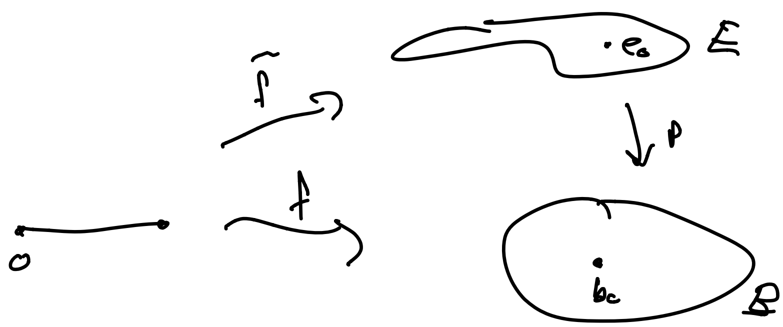
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

$f = p \circ \tilde{f}$

BASIC LIFTING LEMMA Let $p: E \to B$ be a covering space, $e_0 \in E$ a basepoint & $b_0 = p(e_0) \in B$. Let $f: ([0,1], \{0\}) \to (B, b_0)$ be a continuous map. Then $\exists!$ lift $\tilde{f}: ([0,1], \{0\}) \to (E, e_0)$.



$f = p \circ \tilde{f}$

PROOF The proof is almost exactly the same as when $B = S^1$ & $E = \mathbb{R}$.

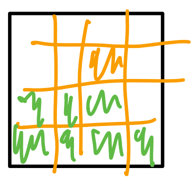
- Find $0 = t_0 < t_1 < \dots < t_n = 1$ s.t. that $f([t_{i-1}, t_i]) \subset U_i$ where U_i is evenly covered.

- Define $\tilde{f}(0) = e_0$
- Assume $\tilde{f}$ is defined on $[0, t_{i-1}]$ & let $\hat{U}_i$ be the component of $p^{-1}(U_i)$ that contains $\tilde{f}(t_{i-1})$.
- Let p_i^{-1} be the inverse of p restricted to $\hat{U}_i$.
- Define $\tilde{f}$ on $[t_{i-1}, t_i]$ by
$$\tilde{f} = p_i^{-1} \circ f.$$

Homotopy Lifting Lemma Let $p:(E,e_0)\to(B,b_0)$ be a covering space & $f:[0,1]\times[0,1]\to B$ a continuous map with $f(0,0)=b_0$. Then $\exists!$ lift $\tilde{f}:[0,1]\times[0,1]\to E$ with $\tilde{f}(0,0)=e_0$.

$f = p\circ\tilde{f}$

PF Again the proof is the same



Key Corollary: if 2 paths have the same endpoints and are path homotopic then their lifts have the same endpoint

COR Let $p: E \to B$ be a covering space & let

$$f, g : [0,1] \to B$$

be paths with $f(0) = g(0)$, $f(1) = g(1)$, and $f \simeq_p g$. If $\tilde{f}, \tilde{g} : [0,1] \to E$ are lifts of f & g with $\tilde{f}(0) = \tilde{g}(0)$ then $\tilde{f}(1) = \tilde{g}(1)$.

PF Let $F: [0,1] \times [0,1] \to B$

be the homotopy between f & g. Let $\tilde{F} : [0,1] \times [0,1] \to E$ be the lift of F with $\tilde{F}(0,0) = \tilde{f}(0) = \tilde{g}(0)$.

$\square$ As F is a homotopy of pairs $\tilde{F}$ is constant on $\{0\} \times [0,1]$ & $\{1\} \times [0,1]$.

Also $\tilde{f}_t(s) = \tilde{F}(s,t)$ is a lift of $f_t(s) = F(s,t)$. Since $\tilde{f}_t(0) = \tilde{F}(0,t) = \tilde{f}(0) = \tilde{g}(0)$ by the uniqueness of lifts $\tilde{f}_0 = \tilde{f}$ & $\tilde{f}_1 = \tilde{g}$.

Since $\tilde{F}$ is constant on $\{1\} \times [0,1]$ we have $\tilde{f}_0(1) = \tilde{f}_1(1) \Rightarrow \tilde{f}(1) = \tilde{g}(1)$. ∎

DEFINITION If X is path connected & $\pi_1(X, x_0) = \{e\}$ then X is *simply connected*.

LEMMA Assume that X is simply connected. Then for any paths

$$f, g : [0,1] \to X$$

with $f(0) = g(0)$ & $f(1) = g(1)$ we have $f \simeq_p g$.

PROOF $f * \bar{g}$ represents an element of $\pi_1(X, f(0))$. Since X is simply connected $f * \bar{g} \simeq_p \text{const}$

$$\Rightarrow \quad f * \bar{g} * g \simeq_p \text{const} * g \simeq_p g$$

But $\bar{g} * g \simeq_p \text{const}$. so

$$f * \bar{g} * g \simeq_p f * \text{const} \simeq_p f.$$

Therefore $f \simeq_p g$. ■

$\boxed{\text{ANOTHER !! LIFTING LEMMA}}$ Assume that X is simply connected and that $p:(E,e_0) \to (B,b_0)$ is a covering space. Fix a basepoint $x_0 \in X$. Then any map

$$f: (X, x_0) \longrightarrow (B, b_0)$$

has a unique lift

$$\tilde{f}: (X, x_0) \longrightarrow (E, e_0).$$

$\boxed{\text{PROOF}}$ Given $x \in X$ define $\tilde{f}(x)$ by choosing a path

$$\alpha: [0,1] \to X$$

with $\alpha(0) = x_0$ & $\alpha(1) = x$, and setting $\tilde{f}(x) = \tilde{\alpha}(1)$ where $\tilde{\alpha}$ is the lift of $f\circ\alpha$. This is well defined since for any other path β with $f\circ\alpha(0) = f\circ\beta(0)$ & $f\circ\alpha(1) = f\circ\beta(1)$ we have $\tilde{\alpha}(1) = \tilde{\beta}(1)$ by lemma.