$\boxed{\text{BASIC LIFTING LEMMA}}$ Let $p: E \to B$ be a covering space, $e_0 \in E$ a basepoint & $b_0 = p(e_0) \in B$. Let $f: ([0,1], \{0\}) \to (B, b_0)$ be a continuous map. Then $\exists!$ lift $\tilde{f}: ([0,1], \{0\}) \to (E, e_0)$.

$\boxed{\text{Homotopy Lifting Lemma}}$ Let $p: (E, e_0) \to (B, b_0)$ be a covering space & $F: [0,1] \times [0,1] \to B$ a continuous map with $f(0,0) = b_0$. Then $\exists!$ $\tilde{F}: [0,1] \times [0,1] \to E$ with $\tilde{F}(0,0) = e_0$ and $F = p \circ \tilde{F}$.

$\boxed{\text{COR}}$ Let $p: E \to B$ be a covering space & let

$$f, g : [0,1] \to B$$

be paths with $f(0) = g(0)$, $f(1) = g(1)$, and $f \simeq_p g$. If $\tilde{f}, \tilde{g}: [0,1] \to E$ are lifts of f & g with $\tilde{f}(0) = \tilde{g}(0)$ then $\tilde{f}(1) = \tilde{g}(1)$.

X is simply connected if X is path connected and $\pi_1(X, x_0) = \{id\}$

ANOTHER!! LIFTING LEMMA Assume that X is simply connected and locally path connected. Let $p: (E, e_0) \to (B, b_0)$ be a covering space. Fix a basepoint $x_0 \in X$. Then any map

$$f: (X, x_0) \longrightarrow (B, b_0)$$

has a unique lift

$$\tilde{f}: (X, x_0) \longrightarrow (E, e_0). \qquad f = p \circ \tilde{f}.$$

PROOF Given $x \in X$ define $\tilde{f}(x)$ by choosing a path

$$\alpha: [0,1] \to X$$

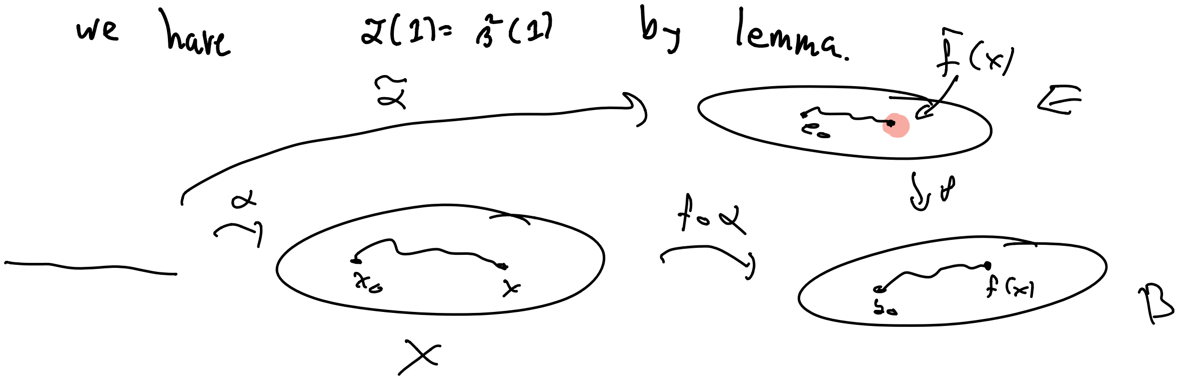
with $\alpha(0) = x_0$ & $\alpha(1) = x$. Then

$$f \circ \alpha: [0,1] \to B$$

is a path to B with $f\circ\alpha(0) = b_0$. By the lifting lemma, $f \circ \alpha$ has a lift

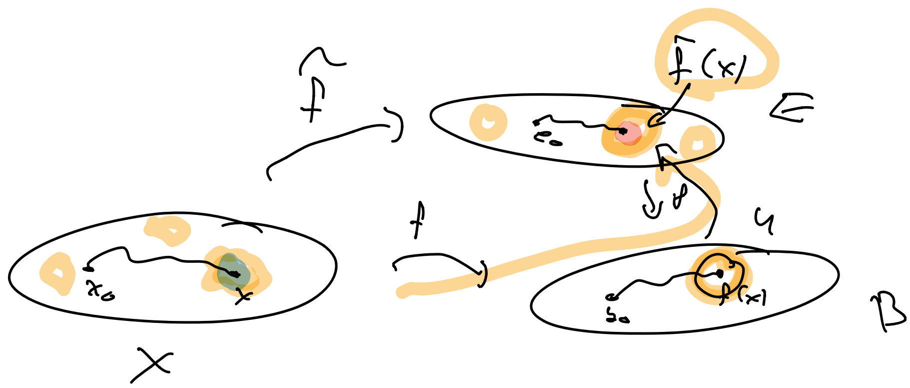
$$\tilde{\alpha}: [0,1] \to E$$

with $\tilde{\alpha}(0) = e_0$ & $f\circ\alpha = p \circ \tilde{\alpha}$. We define $\tilde{f}(x) = \tilde{\alpha}(1)$. This is well defined since for any other path β with $f\circ\alpha(0) = f\circ\beta(0)$ & $f\circ\alpha(1) = f\circ\beta(1)$ we have $\tilde{\alpha}(1) = \tilde{\beta}(1)$ by lemma.



QUESTION Does path connected imply locally path connected?

To prove continuity we need to use that X is locally path connected. That is $\forall x \in X$, and all nbds U of x, there is a path connected nbd V of x with $V \subset U$.



$f^{-1}(U)$ is a nbd of $x \in X$

$V \subset f^{-1}(U)$ that is path connected

Let U be an evenly covered nbd. of $f(x)$. Then $f^{-1}(U)$ is a nbd of x in X and there is a path connected nbd V of x with $V \subset f^{-1}(U)$. Let U_α be the component of $p^{-1}(U)$ that contains $\tilde{f}(x)$. Let p_x^{-1} be the inverse of the restriction of p to U_α. We claim that $\tilde{f} = p_x^{-1} \circ f$ on V. Given $y \in V$ let

$$\beta : [0,1] \to V \subset X$$

be a path with $\beta(0) = x$ & $\beta(1) = y$. Note that $p(\tilde{f}(x)) = f(x)$ so we can apply the lifting lemma to $f \circ \beta$ to find a lift $\tilde{\beta} : [0,1] \to E$ with $\tilde{\beta}(0) = \tilde{f}(x)$.

We can also define a lift of $f \circ \beta$ by taking $p_x^{-1} \circ f \circ \beta$. As $\pi_x^{-1} \circ f \circ \beta(0) = \tilde{f}(x)$ the uniqueness of lifts implies that $\tilde{\beta} = p_x^{-1} \circ f \circ \beta$.

To define $\hat{f}(y)$ we need a path from x_0 to y. The concatenation $\alpha * \beta$ is such a path so $\hat{f}(y) = \widetilde{\alpha * \beta}(1)$ where $\widetilde{\alpha * \beta}$ is the lift of $f \circ (\alpha * \beta)$. However, the concatenation $\tilde{\alpha} * \tilde{\beta}$ is a (and hence the) lift of $f \circ (\alpha * \beta)$ so $\hat{f}(y) = \widetilde{\alpha * \beta}(1) = \tilde{\alpha} * \tilde{\beta}(1) = \tilde{\beta}(1) = \pi_x^{-1} \circ f \circ \beta(1) = p_x^{-1} \circ f(y)$. ■

Let $p : (E, e_0) \to (B, b_0)$ be a covering space. Then

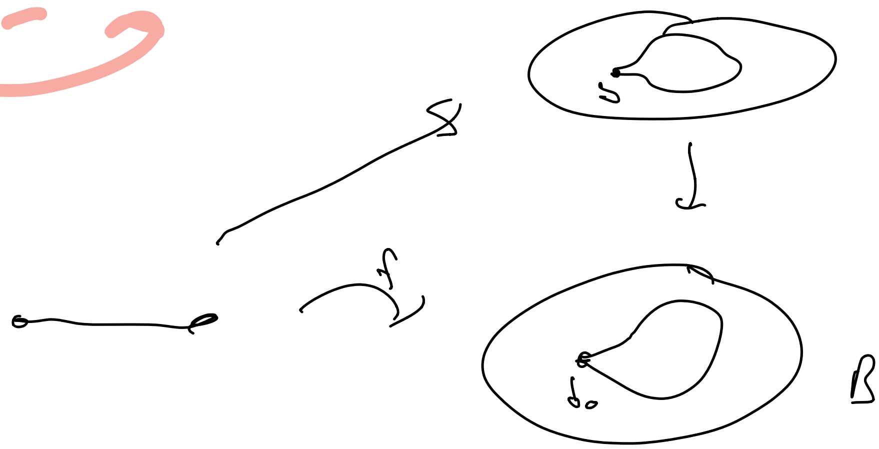
$$p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$$

is the induced homomorphism.

We can apply the lifting lemma to $[f] \in \pi_1(B, b_0)$.

$\boxed{\text{PROPOSITION}}$ If $[f] \in p_*(\pi_1(E, e_0)) \subset \pi_1(B, b_0)$ if $\tilde{f}$ is the lift of f with $\tilde{f}(0) = e_0$ then $\tilde{f}(1) = e_0$.

$\boxed{\text{PROOF}}$ Choose $[g] \in \pi_1(E, e_0)$ such that $p_*([g]) = [f]$. Then $p \circ g \simeq_p f$. By the COR if $\widetilde{p \circ g}$ is the lift of $p \circ g$ then $\widetilde{p \circ g}(1) = \tilde{f}(1)$. But the (unique) lift of $p \circ g$ is g so $g(1) = \tilde{f}(1) = e_0$. ∎



PROPOSITION p_* is injective.

PROOF Assume that $[f] \in \pi_1(E, e_0)$ with $p_*([f]) = id$. Then there is a homotopy of pairs

$$F : [0,1] \times [0,1] \longrightarrow B$$

from $p \circ f$ to id. By the homotopy lifting lemma there is a lift

$$\tilde{F} : [0,1] \times [0,1] \longrightarrow E$$

of F with $\tilde{F}(0,0) = e_0$. This is a homotopy of pairs from f to the id, so $[f] = id$. ∎

$\boxed{\text{FINAL LIFTING LEMMA}}$ Assume X is locally path connected,

$$p : (E, e_0) \to (B, b_0)$$

&

$$f : (X, x_0) \to (B, b_0)$$

with

$$f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0)) \subset \pi_1(B, b_0).$$

Then $\exists!$ lift

$$\tilde{f} : (X, x_0) \to (E, e_0).$$