THE LIFTING CORRESPONDENCE

$p: E \to B$ covering space, E simply connected

$b_0 \in B$, $e_0 \in p^{-1}(b_0)$

We have a bijective map
$$\phi: \pi_1(B, b_0) \to p^{-1}(b_0).$$

Lifting correspondence

$[f] \in \pi_1(B, b_0)$

$$\begin{tikzcd}
 & (E, e_0) \arrow[d, "p"] \\
([0,1], 0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (B, b_0)
\end{tikzcd}$$

Define $\phi([f]) = \tilde{f}(1)$.

ϕ is well defined and a bijection.

Let $e_1 \in p^{-1}(b_0)$. Choose
$\tilde{f}: [0,1] \to E$ s.t. $\tilde{f}(0) = e_0$, $\tilde{f}(1) = e_1$

Let $f = p \circ \tilde{f}$, $[f] \in \pi_1(B, b_0)$

with $\phi([f]) \overset{!}{=} e_1$

Φ is injective

$[f], [g] \in \pi_1(B, b_0)$ with $\phi([f]) = \phi([g])$

need to show $[f] = [g] \iff f \simeq_p g$.

$$\tilde{f}(1) = \tilde{g}(1)$$

Also $\tilde{f}(0) = \tilde{g}(0)$ by def'n of lift.

$$\tilde{f} \simeq_p \tilde{g} \implies f \simeq_p g$$

GROUP ACTIONS

Let X be a simply connected topological space. Then the set of self homeomorphisms, $\text{homeo}(X)$, is a group with group operation composition.

1) $f, g \in \text{homeo}(X) \Rightarrow f \circ g \in \text{homeo}(X)$
2) If $id : X \to X$ is the identity map then $id \in \text{homeo}(X)$ & $f \circ id = id \circ f = f \quad \forall f \in \text{homeo}(X)$.
3) If $f \in \text{homeo}(X)$ then $f^{-1} \in \text{homeo}(X)$ & $f \circ f^{-1} = f^{-1} \circ f = id$.
4) Composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.

A subgroup $G \subset \text{homeo}(X)$ is a group action on X. Let

$$Gx = \{ y \in X \mid \exists g \in G \text{ with } g(x) = y \}$$

be the G-orbit of $x \in X$. We define $x \sim_G y$ if they are in the same G-orbit and let $X/G = X/\sim_G$ with the quotient topology.

1) $x \sim_G x$ since $id \in G$ and $id(x) = x$.
2) $x \sim_G y \Leftrightarrow y \sim_G x$ since if $y = g(x)$ for $g \in G$ then $g^{-1} \in G$ & $x = g^{-1}(y)$.
3) If $x \sim_G y$ & $y \sim_G z$ then $x \sim_G z$ since if $g, h \in G$ with $x = g(y)$ & $z = h(y)$ then

$h \circ g \in G$ & $z = h \circ g(x)$.

G is a deck action if every $x \in X$ has a nbd U such $U \cap g(U) = \phi$ if $g \neq id$.

$\boxed{\text{LEMMA}}$ If $G \subset \text{homeo}(X)$ is a deck action the quotient map $q: X \to X/G$ is a covering map.

$\boxed{\text{PROOF}}$ Since G is a deck action for every $x \in X$ $\exists$ a nbd U s.t. $U \cap gU \neq \phi$ if $g \neq id$. Note that U & gU are homeomorphic since any homeo restricted to a subspace is a homeomorphism onto its image. Furthermore $q(U)$ is a nbd of $q(x) \in X/G$ since quotient maps are open. Then $q(U)$ is an evenly covered nbd of $q(x)$ since

$$\coprod_{g \in G} g(U)$$

is a partition of $q^{-1}(q(U))$ into open sets & q restricted to $g(U)$ is a homeomorphism to $q(U)$. ∎