MIDTERM FRIDAY 2/27 IN CLASS

I'll be out of town - leaving the dept at 2 on Wed.

Topics for midterm

- homotopies, path homotopies
- the fundamental group
- covering spaces
- lifting lemmas

midterm

- true / false example / counterexample questions
  - If $f \simeq g$ & $g \simeq h$ then $f \simeq h$.
  - If $[f], [g], [h] \in \pi_1(X, x_0)$ with
  - Give an example of a covering space $p: (E, e_0) \to (B, b_0)$ & a map $f: (X, x_0) \to (B, b_0)$ s.t. f doesn't have a lift.

    Using the same spaces find a map $g: (X, x) \to (B, b_0)$ that does have a lift.

2 short proofs - I'll give you 5 to choose from.

RETRACTIONS X is a top. space, $A \subset X$ subspace
$r: X \to A$ is a *retraction* if
$r|_A = id$. That is $r(x) = x \quad \forall x \in A$.

THEOREM Let $r: X \to A$ be a retraction and
$\iota_A: A \to X$ the inclusion map. Then
1) $r_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.
2) $(\iota_A)_*: \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.

PROOF Let $id_A: A \to A$ be the identity map.
Then $id_A = r \circ \iota_A$ and therefore $(id_A)_* = r_* \circ (\iota_A)_*$.

GENERAL FACT If $f \circ g = h$ & h is injective
then g is injective (but not necessarily f).

If $f \circ g = h$ & h is surjective then
f is surjective (but not necessarily g).

The theorem follows directly from the general fact! $\blacksquare$

COROLLARY Let X be a topological space &
$S^1 \subset X$ a subspace. Let $x_0 \in S^1 \subset X$ be a basepoint.
If $\pi_1(X, x_0) = \{1\}$ there is no retraction
from X to S^1.

PROOF Let $r: X \to S^1$ be a retraction. Then r_* is
surjective but $\pi_1(X, x_0) = \{1\}$ while $\pi_1(S^1, x_0) = \mathbb{Z}$. Contradiction. $\blacksquare$

Let $B^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ and $x_0 \in B^2$ a basepoint.

Then $\pi_1(B^2, x_0) = \{1\}$. PROOF: Straight line homotopy.

If $[f] \in \pi_1(B^2, x_0)$ define

$$f_t(s) = (1-t) f(s) + t \cdot x_0$$

$\Rightarrow f \simeq_p$ constant map to x_0.

COROLLARY There is no retraction $r: B^2 \to S^1$.

PROOF Follows from previous corollary since $\pi_1(B^2) = \{1\}$.

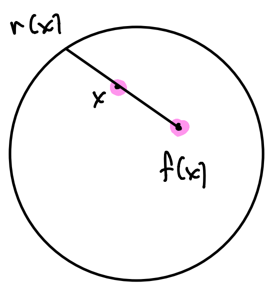
BROUWER FIXED POINT THEOREM IN DIMENSION 2

Every map $f: B^2 \to B^2$ has a fixed point.

($\exists\, x \in B^2$ s.t. $f(x) = x$.)

PROOF We'll show that if f doesn't have a fixed point then there is a retraction r from B^2 to S^1.

The construction:



A formula:

$\forall\, x \in B^2$ choose $s_x \in (0,2]$ s.t.

$$|(1-s_x) f(x) + s_x \cdot x| = 1$$

& define

$$r(x) = (1-s_x) f(x) + s_x \cdot x.$$

Note that if $x \in S^1$, $s_x = 1$ & $r(x) = x$.

Why is this map continuous?