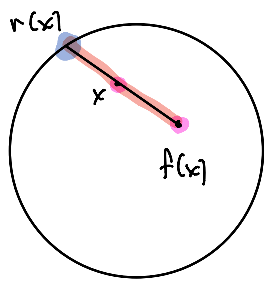
THM Let $f: B^2 \to B^2$ be continuous. Then $\exists\, x \in B^2$ s.t. $f(x) = x$.

PROOF We'll show if f doesn't have a fixed point $\Rightarrow \exists$ a retraction $r: B^2 \to S^1$. $\Rightarrow$ Contradiction

The construction:



A formula:

$\forall\, x \in B^2$ choose $s_x \geq 1$ s.t.

$$|(1-s_x) f(x) + s_x \cdot x| = 1$$

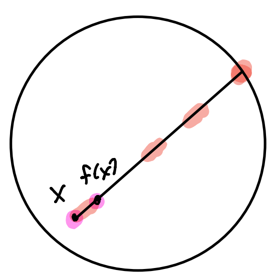
& define

$$r(x) = (1-s_x) f(x) + s_x \cdot x.$$

Note that if $x \in S^1$, $s_x = 1$ & $r(x) = x$.

Why is this map continuous?

Note that if $|x - f(x)|$ is very small then s_x can be very large.



s_x is the ratio $\dfrac{|r(x) - f(x)|}{|f(x) - x|}$

Note that $|r(x) - f(x)| \leq 2$.

If we can bound $|f(x) - x|$ from below we bound s_x from above.

$x \mapsto |f(x) - x|$ is a continuous fn on a compact set B^2. Let $\varepsilon = \inf_{x \in B_2} |f(x) - x|$.

As B^2 is compact $\exists\, x_0 \in B^2$ s.t. $\varepsilon = |f(x_0) - x_0| > 0$.

$\Rightarrow \quad 1 \leq s_x \leq \frac{2}{2} = \delta \quad \forall x \in B^2.$

This is where we use that $f(x) \neq x$.

Define $R: B^2 \times [1, \delta] \longrightarrow \mathbb{R}^2$ by

$$R(x, s) = (1-s) f(x) + s \cdot x .$$

R is continuous by standard theorems on continuity.
Then $R^{-1}(S^1)$ is closed in $B^2 \times [1, \delta]$ since the pre-image of closed sets are closed under continuous maps.

For each $x \in B^2$ the equation $|R(x,s)| = 1$ has a unique solution in $[1, \delta]$.

So we have a bijection

$$\sigma : B^2 \to R^{-1}(S^1)$$

with

$$R \circ \sigma(x) = r(x).$$

We need to show that σ is continuous.
Let $K \subset R^{-1}(S^1)$ be closed as a subspace of $R^{-1}(S^1)$.
As $R^{-1}(S^1)$ is closed in $[1, \delta]$, K is also closed in $B^2 \times [1, \delta] \Rightarrow K$ is compact.

The projection

$$q : B^2 \times [1, \delta] \to B^2$$

with $q(x,s) = x$ is continuous so $q(K)$ is compact & therefore closed. As $\sigma^{-1}(K) = q(K)$ the σ-pre-image of closed sets is closed so σ is continuous. $\blacksquare$

LEMMA If $f, g : (X, x_0) \to (Y, y_0)$ are homotopic as pairs then $f_* = g_*$.

PROOF Let $[h] \in \pi_1(X, x_0)$ then
$f_*[h] = [f \circ h]$ & $g_*([h]) = [g \circ h]$
As f & g are homotopic as pairs so are $f \circ h$ & $g \circ h$ $\Rightarrow$ $[f \circ h] = [g \circ h]$. ∎
are also homotopic as pairs.

LEMMA Let $f_t : X \to Y$ be a homotopy & let $x_0 \in X$ be a basepoint. Let
$\alpha(t) = f_t(x_0)$ be a path from
$y_0 = f_0(x_0)$ & $y_1 = f_1(x_0)$. $\Rightarrow \hat{\alpha} : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$
Then $\hat{\alpha} \circ (f_0)_* = (f_1)_*$ where
$(f_0)_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$
& $(f_1)_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$

PROOF Define $\alpha_t : [0,1] \to Y$ by
$$\alpha_t(s) = f_t(x_0).$$
Given $h \in \pi_1(X, x_0)$ define
$$h_t = \alpha_t * (f_t \circ h) * \bar{\alpha}_t.$$
h_t is a homotopy from
$$\bar{\alpha}_0 * (f_0 \circ h) * \alpha_0 \simeq_p [f_0 \circ h] = (f_0)_*([h]) \quad \text{to}$$
$$\bar{\alpha}_1 * (f_1 \circ h) * \alpha_1 = \bar{\alpha} * (f_1 \circ h) * \alpha = \hat{\alpha}^{-1} \circ (f_1)_*([h]).$$ ∎

$\boxed{\text{THE FUNDAMENTAL THEOREM OF ALGEBRA}}$

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ be a polynomial with $a_n \neq 0$ & $n \geq 1$. Then $\exists z \in \mathbb{C}$ s.t. $p(z) = 0$.

$\boxed{\text{PROOF}}$ Assume that $p(z) \neq 0$ $\forall z \in \mathbb{C}$.

Let $\sigma : [0,1) \to [0,\infty)$ be a homeomorphism and define

$$f_t : S^1 \to S^1$$

by

$$f_t(z) = \frac{p(\sigma(t) z)}{|p(\sigma(t) z)|} \qquad 0 \leq t < 1$$

Then $f_0 \equiv \frac{p(0)}{|p(0)|}$.

f_1 is not defined initially but we can define

$$f_1(z) = \lim_{t \to 1} \frac{p(\sigma(t) z)}{|p(\sigma(t) z)|}$$

$$= \lim_{t \to 1} \frac{a_n(\sigma(t) z)^n + a_{n-1}(\sigma(t) z)^{n-1} + \dots + a_0}{|a_n(\sigma(t) z)^n + a_{n-1}(\sigma(t) z)^{n-1} + \dots + a_0|}$$

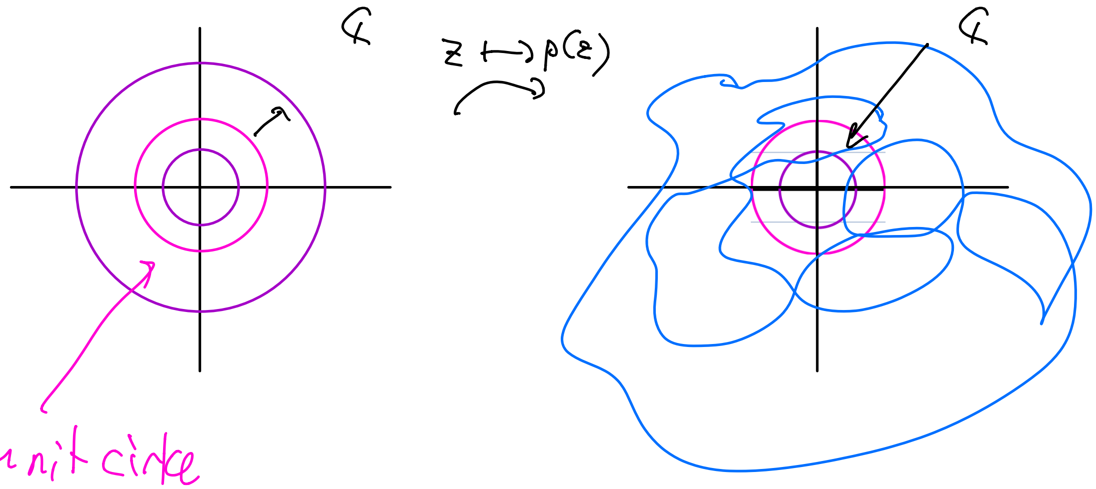
As $t \to 1$
$\sigma(t) \to \infty$

$$= \lim_{t \to 1} \frac{a_n z^n + a_{n-1} \frac{z^{n-1}}{\sigma(t)} + a_{n-1} \frac{z^{n-2}}{\sigma(t)^2} + \dots + \frac{a_0}{\sigma(t)^n}}{\left|a_n z^n + a_{n-1} \frac{z^{n-1}}{\sigma(t)} + a_{n-1} \frac{z^{n-2}}{\sigma(t)^2} + \dots + \frac{a_0}{\sigma(t)^n}\right|}$$

$$= \frac{a_n z^n}{|a_n z^n|} = \frac{a_n}{|a_n|} \frac{z^n}{|z^n|} = \frac{a_n}{|a_n|} z^n$$

$(f_0)_*([u]) = 0$ $\qquad$ $(f_1)_*([u]) = n \cdot [u]$

$\Rightarrow$ contradiction ∎


ℂ
$z \mapsto p(z)$
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unit circle