$$\pi : \mathbb{R} \to \mathbb{R}/\sim \; = S^1$$

$t, t' \in \mathbb{R}$ $\quad t \sim t'$ $\quad$ if $\quad t = t' + n$, $n \in \mathbb{Z}$

$\pi(t) = [t]$, $\quad [t]$ is the equivalence class

LIFTING LEMMA Let $f : ([0,1], \{0\}) \to (S^1, [0])$.

$$\exists! \; \tilde{f} : ([0,1], \{0\}) \to (\mathbb{R}, \{0\})$$

with $\quad f = \pi \circ \tilde{f}$.

$\boxed{\tilde{f} \text{ is a lift of } f}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ ([0,1], 0) \arrow[ur, squiggly, "\tilde{f}"] \arrow[r] & (S^1, [0]) \end{tikzcd}$$

Homotopy Lifting Lemma

Let $F : ([0,1]\times[0,1], \{(0,0)\}) \to (S^1, [0])$.

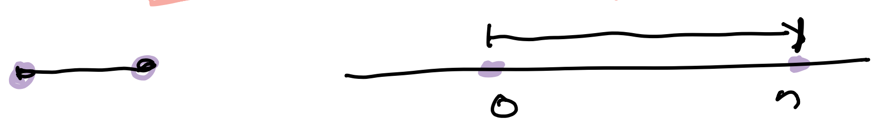
$\exists!\ \tilde{F} : ([0,1]\times[0,1], \{(0,0)\}) \to (\mathbb{R}, \{0\})$

with $F = \pi \circ \tilde{F}$.

Recall maps $\tilde{f}_n : ([0,1], \{0,1\}) \to (\mathbb{R}, \mathbb{Z})$

and $f_n : ([0,1], \{0,1\}) \to (S^1, [0])$

with $\tilde{f}_n(t) = nt$ and $f_n = \pi \circ \tilde{f}_n$.



THM Given $f : ([0,1], \{0,1\}) \to (S^1, [0])$

there exists a unique $n \in \mathbb{Z}$ s.t.

$f \simeq_p f_n$.

$\boxed{\text{PF}}$ We have already seen that there exists an $n \in \mathbb{Z}$ s.t $f \simeq_p f_n$.

$\boxed{\text{QUICK REVIEW}}$ By the L.L $\exists$ a lift

$$\tilde{f} : ([0,1], \{0\}) \to (\mathbb{R}, \{0\})$$

$$\begin{tikzcd}
 & (\mathbb{R}, \{0\}) \arrow[d, "\pi"] \\
([0,1], \{0\}) \arrow[ur, "\tilde{f}", "L.L"'] \arrow[r, "f"'] & (S^1, [0])
\end{tikzcd}$$

As $f(1) = \pi \circ \tilde{f}(1) = [0] \Rightarrow \tilde{f}(1) \in \mathbb{Z}$

and $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$.

Then we claim $f \simeq_p f_n$.

Define $\tilde{G} : [0,1] \times [0,1] \to \mathbb{R}$ — straight line homotopy

by $\tilde{G}(s,t) = (1-t)\tilde{f}(s) + t\tilde{f}_n(s)$

and let $G(s,t) = \pi \circ \tilde{G}(s,t)$. — homotopy between f & f_n

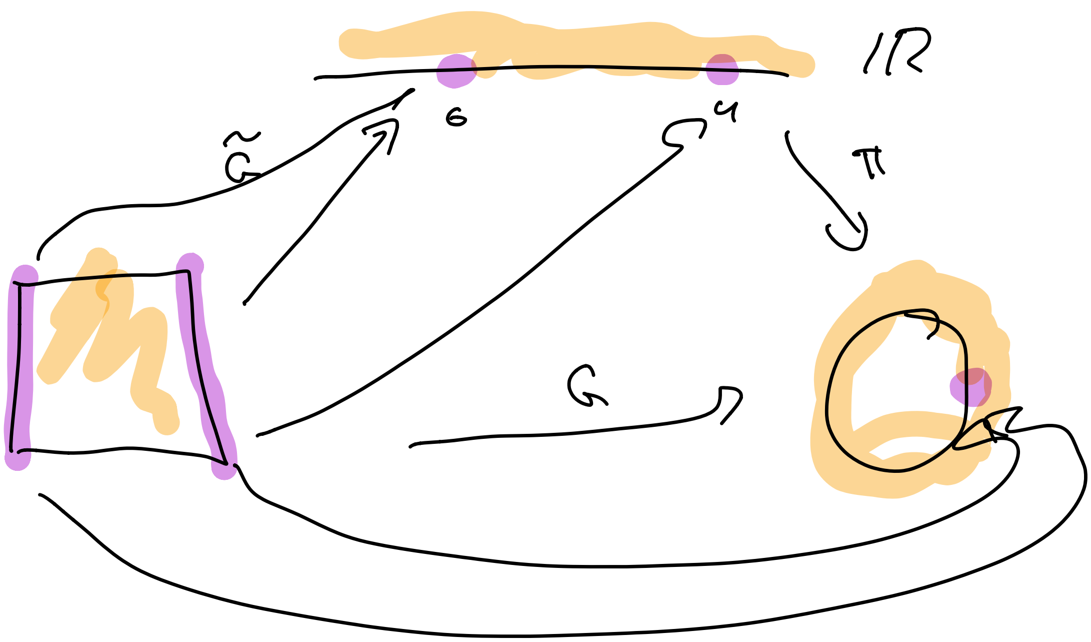
Note $\tilde{g}_t(0) = (1-t)\,\underset{0}{\underset{\shortparallel}{\tilde{f}(0)}} + t\,\underset{0}{\underset{\shortparallel}{\tilde{f}_n(0)}} = 0$ & need $f \simeq_p f_1$

$n = \tilde{g}_t(1) = (1-t)\,\underset{n}{\underset{\shortparallel}{\tilde{f}(1)}} + t\,\underset{n}{\underset{\shortparallel}{\tilde{f}(1)}} = (1-t)n + tn = n$

so $\tilde{G}(\{0\}\times[0,1]) \subset \{0\}$ & $\tilde{G}(\{1\}\times[0,1]) \subset \{n\}$.

$\Rightarrow$ $G(\{0,1\}\times[0,1]) = [0]$.

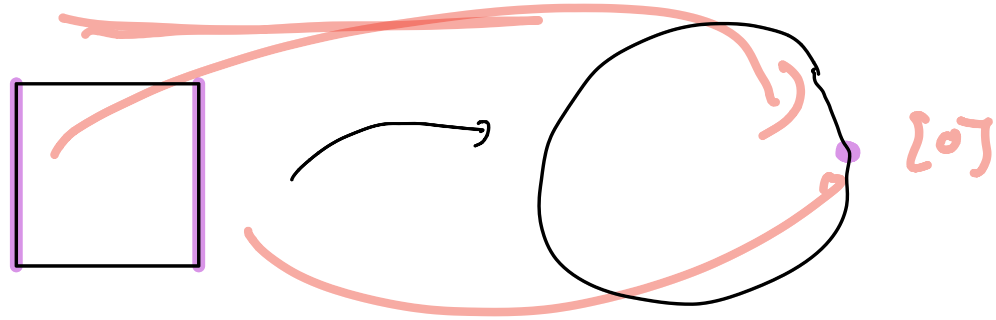
$\Rightarrow$ $f \simeq_p f_1$.



$\boxed{\text{NOW WE SHOW THAT IF } f \simeq_p f_m \text{ THEN } n = m.}$

Let $G : [0,1] \times [0,1] \to S^1$ be the homotopy realizing $f \simeq_p f_m$.

Then $G(\{0,1\} \times [0,1]) = [0]$.



In particular $G(0,0) = [0]$.

By the H.L.L. $\exists!$ lift

$\tilde{G} : ([0,1] \times [0,1], (0,0)) \to (\mathbb{R}, \{0\})$.

$\tilde{g}_t(s) = \tilde{G}(s,t)$

$G = \pi \circ \tilde{G}$

Note that $\tilde{g}_1$ is a lift of f_m. $g_t = \pi \circ \tilde{g}_t$

$t = 1$.

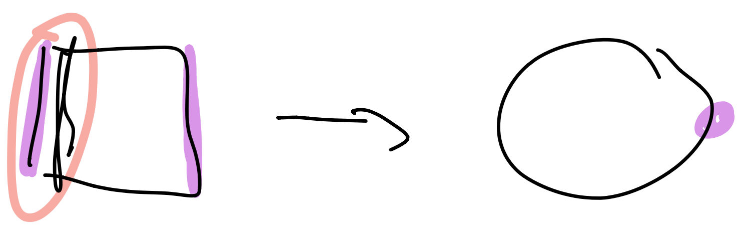
By the uniqueness of lifts $\tilde{g}_1 = \tilde{f}_m$ $\tilde{g}_1(0) = 0$

Since $\tilde{f}_m$ is also a lift of f_m.

$$\begin{tikzcd}
 & & (\mathbb{R}, \{0\}) \arrow[d, "\pi"] \\
([0,1], \{0\}) \arrow[urr, "\tilde{G}"] \arrow[rr, "G"] & & (S^1, [0]) \\
1 \arrow[urr, bend right] & &
\end{tikzcd}$$

H.L.L.

$$\pi \circ \tilde{g}_t(0) = g_t(0) = [0]$$

$G:$ 

$$\Rightarrow \quad \tilde{g}_t(0) \subset \mathbb{Z} = \pi^{-1}([0])$$

discrete

$$\Rightarrow \tilde{g}_t(0) = \tilde{g}_0(0) = 0$$

$$\Rightarrow \tilde{g}_1(0) = 0$$

$$\tilde{g}_0 \simeq_p \tilde{g}_1 \simeq_p \tilde{f}_m$$
$$\simeq_p \tilde{f}_n$$

$([0,1], \{0,1\}) \to (\mathbb{R}, \mathbb{Z})$

$\simeq$ is with respect to

$\Rightarrow \tilde{f}_n \simeq_p \tilde{f}_m \Rightarrow n = m.$

We've proved:

$\boxed{\text{THM}}$ Given $f : ([0,1], \{0,1\}) \to (S^1, [0])$

there exists a unique $n \in \mathbb{Z}$ s.t.
$$f \simeq_p f_n.$$

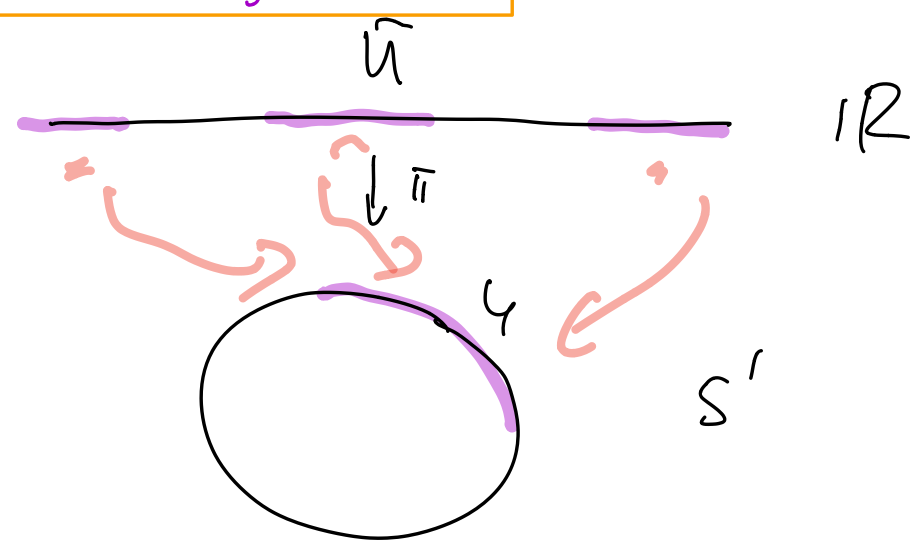
We need to prove the 2 lifting lemmas. First we need:

$\boxed{\text{LEMMA}}$ Every $x \in S^1$ has a nbd U s.t that if $\tilde{U}$ is a connected component of $\pi^{-1}(U)$ then $\pi|_{\tilde{U}}$ is a homeomorphism to U.

$\subset \mathbb{R}$

$\boxed{U \text{ is evenly covered.}}$

$\boxed{\text{PF}}$



- If $t \neq t' \in \mathbb{R}$ & $\pi(t) = \pi(t')$ then $t = t' + n$, $n \in \mathbb{Z} \setminus \{0\}$

$\Rightarrow$ $|t - t'| \geq 1$.

- The closure of any bounded set in $\mathbb{R}$ is compact. Heine-Borel

- If $K \subset \mathbb{R}$ is compact and $\operatorname{diam} K < 1$ then $\pi|_K$ is a homeomorphism onto its image

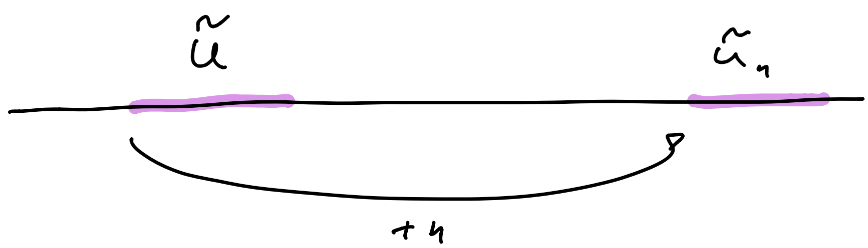
- If $A \subset K$ then $\pi|_A$ is a homeomorphism onto its image

$\Rightarrow$ if $\tilde{u}$ is an open interval in $\mathbb{R}$ of width < 1 then $\pi|_{\tilde{u}}$ is a homeo. onto its image.

Let $U = \pi(\tilde{U})$. Note every $x \in S^1$ is contained in such a U.

What is $\pi^{-1}(U)$? $\Rightarrow$ $\pi^{-1}(U) = \{\tilde{U}_n\}$

Let $\tilde{U}_n$ be the translate of $\tilde{U}$ by n.



Then $\pi(\tilde{U}_n) = U$ &

$\pi|_{\tilde{U}_n}$ will be a homeomorphism to U.

$\blacksquare$

$\boxed{\text{LIFTING LEMMA}}$ Let $f : ([0,1], \{0\}) \to (S^1, [0])$.

$\exists!\ \tilde{f} : ([0,1], \{0\}) \to (\mathbb{R}, \{0\})$

with $f = \pi \circ \tilde{f}$.

$\boxed{\text{Pf}}$ We need to find a partition

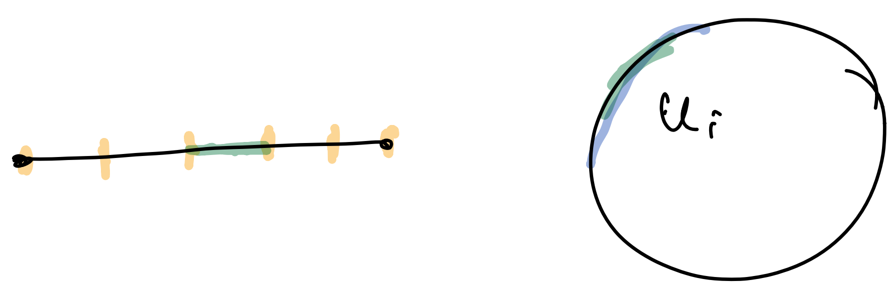
$t_0 = 0 < t_1 < \dots < t_n = 1$ $[0,1]$

of $[0,1]$ s.t. that for

each interval $[t_i, t_{i+1}]$ $\exists$ an

evenly covered nbd. $U_i \subset S^1$

with $f([t_i, t_{i+1}]) \subset U_i$.



Now assume $\tilde{f}$ is defined on $[0,t_i]$. As $f(t_i) \subset U_i$, we have $\tilde{f}(t_i) \in \pi^{-1}(U_i)$.

$f([t_i,t_{i+1}]) \subset U_i$

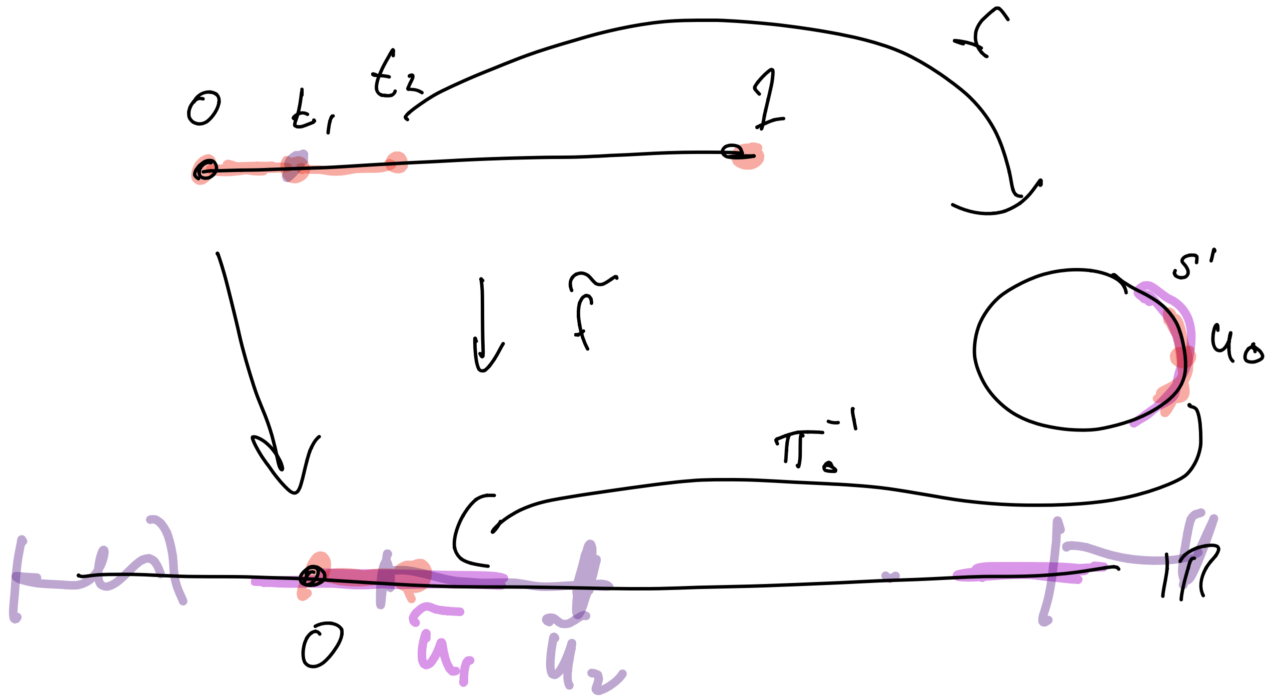
note if $i=0$ $\tilde{f}(0)=0$

Let $\tilde{U}_i$ be the component of $\pi^{-1}(U_i)$ that contains $\tilde{f}(t_i)$.

π_i^{-1} is the inverse of $\pi|_{\tilde{U}_i}$.

Extend $\tilde{f}$ to $[t_i, t_{i+1}]$ by

$$\tilde{f}|_{[t_i,t_{i+1}]} = \pi_i^{-1} \circ f.$$



0
t_1
t_2
1
γ
$\tilde{f}$
S^1
u_0
π_0^{-1}
0
$\tilde{u}_1$
$\tilde{u}_2$
$\mathbb{R}$