Given $f : ([0,1], \{0,1\}) \to (X, \{x_0\})$

we let $[f]$ be the equivalence class

of maps homotopic to f as pairs.

As a set $\pi_1(X, x_0)$ is equivalence classes $[f]$.

We've defined $f * g$.

not initially defined on equivalence classes

We need to show that

$[f] * [g] = [f * g]$ is well defined.

That is we need to show

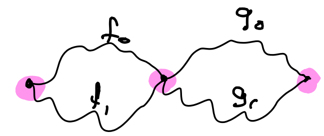
if $f_0 \simeq_p f_1$ & $g_0 \simeq_p g_1$

then $[f_0 * g_0] = [f_1 * g_1]$.

Let F, G be the two

homotopies (between f_0 & f_1 and

between g_0 & g_1).



Define

$$H(s,t) = \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s-1, t) & 1/2 < s \leq 1 \end{cases}$$

H is a homotopy between $f_0 * g_0$

& $f_1 * g_1$

$\Rightarrow$ $[f_0 * g_0] = [f_1 * g_1]$.

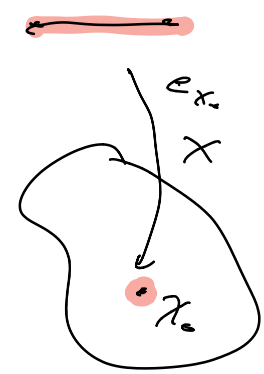
$\boxed{\text{CLOSURE}}$

IDENTITY

Need to find a map

$$e_{x_0} : ([0,1], \{0,1\}) \longrightarrow (X, \{x_0\})$$

such that



$$\boxed{f * e_{x_0}} \simeq_p e_{x_0} * f \simeq_p f.$$

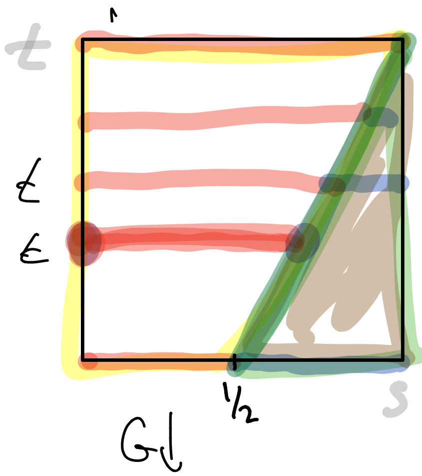
The constant map!

$$e_{x_0}(t) = x_0.$$

Again $f * e_{x_0} \neq f$ but only

$$f * e_{x_0} \simeq_p f.$$

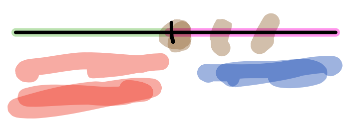
Again need to reparameterize:



$s=0$ $\quad 0 \quad$ at $s = \frac{1+\epsilon}{2}$, $\frac{1}{2}$

$$G(s,t) = \begin{cases} \frac{s}{1+t} & \text{if } s \le \frac{1+t}{2} \\ 1/2 & s > \frac{1+t}{2} \end{cases}$$

$$s = (1-t)\frac{1}{2} + t = \frac{1}{2} + \frac{t}{2}$$



$$H(s,t) = (f * e_{x_0}) \circ G(s,t)$$

$h_0 = f * e_{x_0}$

$h_1 = f$

Also need to check that $H(0,t)$
$= H(1,t) = x_0$

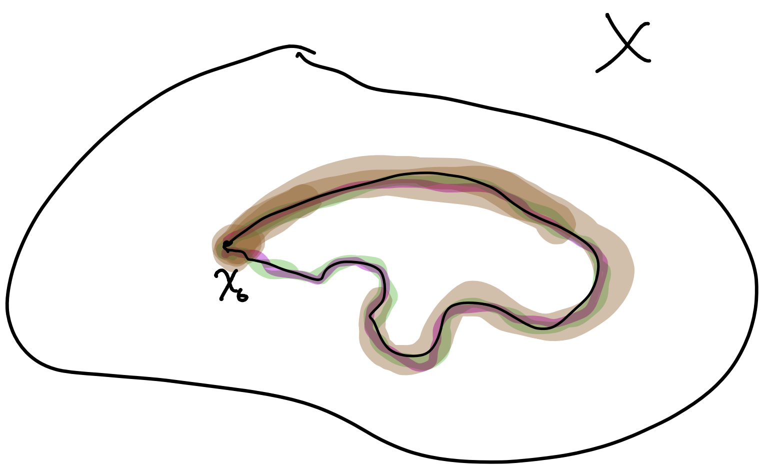
INVERSE Given $f : ([0,1], \{0,1\}) \to (X, \{x_0\})$

need to find $\bar{f}$ s.t

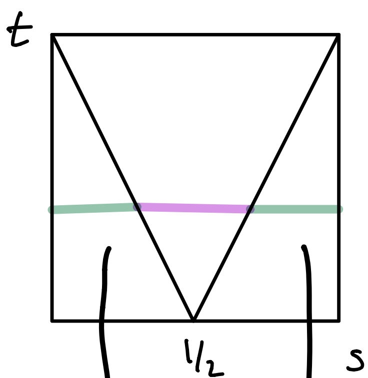
$f * \bar{f} \simeq_p \bar{f} * f \simeq_p e_{x_0}$.

DEFINE: $\bar{f}(t) = f(1-t)$

We'll "wind back" f.



$$f * \bar{f}\,(t) = \begin{cases} f(2t) & t \le 1/2 \\ \bar{f}(2t-1) & t > 1/2 \end{cases}$$



$t = 1 - 2s$

$t = 2s - 1$

Pause on purple

$$f_t(s) = \begin{cases} f(2s) & s \le \frac{1-t}{2} \\ f\left(\frac{1-t}{2}\right) & \frac{1-t}{2} < s \le \frac{t+1}{2} \\ \bar{f}(2s-1) & \frac{t+1}{2} < s \end{cases}$$

STILL NEED TO PROVE ASSOC!

$$\underset{1/4}{(f} * \underset{1/4}{g)} * \underset{1/2}{h} \simeq_p \underset{1/2}{f} * (\underset{1/4}{g} * \underset{1/4}{h})$$

$\boxed{\text{The fundamantel group of } S^1}$

We have done all the work to calculate $\pi_1(S^1, [0])$.

An element $[f]$ of $\pi_1(S^1,[0])$ is an equivalence class of maps

$$f : ([0,1], \{0,1\}) \to (S^1, [0])$$

We have seen that $[f] = [f_n]$ for a unique $n \in \mathbb{Z}$. This defines a map

$$\phi : \pi_1(S^1,[0]) \longrightarrow \mathbb{Z} \qquad \text{by} \qquad \phi([f]) = n$$

where n is the unique integer such that $[f] = [f_n]$.

Clearly $\phi([f_n]) = n$ so this map is surjective. By uniqueness it is also injective $\Rightarrow$ the map is a bijection.

$0 < n < M$
on Wed

We have also seen that $[f_n * f_m] = [f_{n+m}]$ so

$$\phi([f_n * f_m]) = \phi([f_{n+m}]) = n+m = \phi([f_n]) + \phi([f_m]).$$

Therefore ϕ is a homomorphism. A homomorphism that is a bijection is an isomorphism.

What is the l.f.t of $f_n * f_m$?

$$\begin{tikzcd} & (\mathbb{R}, 0) \arrow[d, "\pi"] \\ ([0,1], 0) \arrow[ur, "\widetilde{f_n * f_m}\,?"] \arrow[r, "f_n * f_m"'] & (S^1, [0]) \end{tikzcd}$$

What is $\tilde{f}_n(1) = ?$

n!

But $\tilde{f}_m(0) = 0$

maybe $\widetilde{f_n * f_m} = \cancel{\tilde{f}_n * \tilde{f}_m}$

$$\widetilde{f_n * f_m} = \begin{cases} \tilde{f}_n(2t) & t \leq 1/2 \\ \tilde{f}_m(2t-1) + n & t > 1/2 \end{cases}$$

$\widetilde{f_n * f_m}$ is continuous since
$\tilde{f}_n(2(1/2)) = \tilde{f}(2(1/2)-1) + n = n$

$\widetilde{f_n * f_m}$ is a lift since
$f_n * f_m = \pi \circ \widetilde{f_n * f_m}$

$\widetilde{f_n * f_m}(0) = 0$.

What is $\widetilde{f_n * f_m}(1) = \tilde{f}_m(1) + n$
$= m + n$

$\Rightarrow f_n * f_m \simeq_p f_{n+m}$