Name:

Midterm, Math 5520
February 28, 2020

True/False: Let $p\colon (E, e_0) \to (B, b_0)$ be a covering space between locally path connected spaces. Indicate whether the following statements are true or false. No explanation is necessary.

1. Every map $f\colon ([0,1], 0) \to (B, b_0)$ has a lift $\tilde{f}\colon ([0,1], 0) \to (E, b_0)$. That is $\tilde{f}$ is a continuous map with $f = p \circ \tilde{f}$ and $\tilde{f}(0) = e_0$.

2. Let x be a point in S_1. Every map $f\colon (S^1, x) \to (B, b_0)$ has a lift $\tilde{f}\colon (S^1, x) \to (E, b_0)$. That is $\tilde{f}$ is a continuous map with $f = p \circ \tilde{f}$ and $\tilde{f}(x) = e_0$.

3. Let $\alpha\colon ([0,1], \{0,1\}) \to (B, b_0)$ be a continuous paths. Then there exists a lift $\tilde{\alpha}\colon ([0,1], \{0,1\}) \to (E, e_0)$. That is $\tilde{\alpha}$ is a continuous path with $\alpha = p \circ \tilde{\alpha}$ and $\tilde{\alpha}(0) = \tilde{\alpha}(1) = e_0$.

Examples: For the following two problems you don't need to justify your examples but make sure that you describe them explicitly.

1. Give an example of topological spaces X and Y and maps $f, g\colon X \to Y$ such that f and g are not homotopic.

2. Give an example of a space Y and continuous paths

$$\alpha, \beta\colon [0,1] \to Y$$

with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$ such that α and β are not homotopic as paths.

Short proofs: Choose any two of the following problems. Here you need to carefully justify your work. You should do one of the problems on each of the following two pages. Indicate clearly which problem you are doing.

1. Let X be a topological space and $\mathcal{U}$ an open cover of X. That is $\mathcal{U}$ is a collection of open sets whose union is all of X. Let $f\colon [0,1] \to X$ be a continuous map and show that there is a partition $t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that for each i there is a $U_i \in \mathcal{U}$ with $f([t_{i-1}, t_i]) \subset U_i$.

2. Let $\alpha, \beta\colon [0,1] \to X$ be continuous paths and assume that α is the constant path with $\alpha(t) = \beta(0)$ for all $t \in [0,1]$. Explicitly describe a path homotopy between $\alpha * \beta$ and β.

3. Define
$$S^1 = \{(x,y) \in \mathbb{R}^2\}$$
and the function
$$f\colon S^1 \to S^1$$
by $f(x,y) = (x,-y)$. Let $x_0 = (1,0) \in S^1$ and calculate the induced homomorphism
$$f_*\colon \pi_1\left(S^1, x_0\right) \to \pi_1\left(S^1, x_0\right).$$
Make sure to justify your answer.

4. Let
$$p\colon \mathbb{R} \to S^1$$
be a covering map such that $p(n) = p(m)$ for all $n, m \in \mathbb{Z}$. Let $[f], [g] \in \pi_1(S^1, p(0))$ be elements of the fundamental group represented by closed paths f and g. If $\tilde{f}$ and $\tilde{g}$ are lifts of f and g with $\tilde{f}(0) = \tilde{g}(0) = 0$ explicitly give a formula for the lift $\widetilde{f * g}$ of $f * g$ with $\widetilde{f * g}(0) = 0$. Your answer should be given in terms of $\tilde{f}$ and $\tilde{g}$.

5. Let
$$p\colon (E, e_0) \to (B, b_0)$$
be a covering space with both E and B path connected and locally path connected. If the induced homomorphism
$$p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$$
is an isomorphism show that p is a homeomorphism. (**Hint:** Apply the final lifting lemma to the identity map from B to itself. Noting that the identity map is also a covering map you can also apply the final lifting lemma to lift the map p.)

Problem #

Problem #

Scratchwork