# Stable optimal design for uncertain loading conditions

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**Abstract**

Optimal design problems are usually formulated as problems of minimization of the energy, stored in the design under a prescribed loading. Solutions of these problems are unstable to perturbations of the loading.

We suggest a new min-max formulation of the optimal design problem which has a stable solution. The loading is not prescribed, but only a set of admissible loadings is given. The stable optimal design problem is formulated as minimization of the stored energy of the project under the most unfavorable loading. This most dangerous loading is one that maximizes the stored energy over the class of admissible functions. The problem is reduced to minimization of Steklov eigenvalues. Several stable solutions of various optimal design problems are demonstrated; among them are the optimal structure of a material stable to variations in uniaxial loading, the optimal specific stiffness of an uncertainly loaded beam, and the stable design of an optimal wheel.

**Key words:**
structural optimization, stability, Steklov eigenvalues, optimal composites, minimax.

This paper is dedicated to the memory of Professor Sergey Kozlov.

## 1 Introduction.

In a rather extensive literature on optimal design, major attention is paid to optimization of constructions that are subject to a fixed loading. However, the typical situation for the practical use of an optimal design is different: acting forces are either varying in time, or varying from one sample to another, or are unpredictable. This motivates a reformulation of the problem to account possible variations and uncertainties in loading.

One can foresee a significant change in the reformulated design if the loading is not completely known. Indeed, the optimality requirement forces the structure to concentrate its resistivity against applied loading, since its abilities to resist other loadings are limited. This high sensitivity to the loading restricts the applicability of most optimal designs.

The goal of this paper is to formulate a stable optimal design problem with solution insensitive to small variations in the loadings. We show, that this stable reformulation of the problem leads to optimization of Steklov eigenvalues [2, 9]. Such formulation makes the problem similar to the well investigated problem of optimization of the principle eigenfrequency of the vibration of a construction [10, 11]. We introduce a concept of the compliance of a whole construction that is similar to the compliance of the material. The construction compliance is the energy that corresponds to "the worst" admissible loading, or to the maximal energy produced by a loading from a set of admissible loadings. Then we formulate the optimal design problem for the project which minimizes that maximum, or minimizes the compliance. We end up with the min-max problem that leads to minimization of a non-smooth functional. The approach we use, is similar to one developed for the problem of optimization of boundary excitations in nondestructive testing and electrical tomography [4, 5] and to structural optimization [12]. Several optimal design problems are discussed, and their stable solutions are demonstrated.

## 2 The problem formulation

**The problem of optimal design.** The overall compliance of an elastic construction is characterized by the mechanical work produced by an applied loading. This work is equal to the total energy stored in the loaded construction. It is found from the following variational problem

$$H(\boldsymbol{p}, \boldsymbol{f}) = \min_{\boldsymbol{\sigma} \in \Sigma} \left\{ \int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma}) \right\}, \tag{1}$$

where $W$ is the (doubled) elastic energy

$$W(\boldsymbol{p}, \boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{S}(\boldsymbol{p}, \boldsymbol{x}) : \boldsymbol{\sigma}. \tag{2}$$

$\boldsymbol{\sigma}$ is the stress tensor, the set $\Sigma$ is (see (1), (2))

$$\Sigma = \left\{ \boldsymbol{\sigma} : \quad \nabla \cdot \boldsymbol{\sigma} = 0 \ \text{ in } \Omega, \quad \boldsymbol{\sigma} = \boldsymbol{S}^{-1} (\nabla u)^s, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{f} \ \text{ on } \partial\Omega \right\}. \tag{3}$$

$\boldsymbol{f}$ is the vector of applied boundary forces and $\boldsymbol{u}$ is the vector of deflection, $(\nabla \boldsymbol{u})^s = \boldsymbol{\epsilon}$ is the strain field, that is given by a symmetrized part of the gradient of $\boldsymbol{u}$, $(\nabla \boldsymbol{u})^s = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$. $\boldsymbol{S}(\boldsymbol{p}, \boldsymbol{x})$ is the tensor of elastic compliance: a fourth order symmetric positive tensor, which depends on the point $\boldsymbol{x}$ in $\Omega$ and on the structural parameter $\boldsymbol{p}$ that defines the material's properties. The symbol (:) denotes the contraction by two indices.

The stored energy $H$ is a quadratic functional of the loading $\boldsymbol{f}$ that depends on the layout of the material's properties $\boldsymbol{p}$, called the design variables.

Consider the typical problem of optimal design: minimize $H$ with respect to layout $\boldsymbol{p}$:

$$\min_{\boldsymbol{p} \in \mathcal{P}} \ H(\boldsymbol{p}, \boldsymbol{f}), \tag{4}$$

where $\mathcal{P}$ is the admissible set of design variables. There are many possible settings for the set $\mathcal{P}$ : it can be defined as the set of effective moduli of the composite [8], or it could describe the shape of the body, the thickness of a thin construction, and so on.

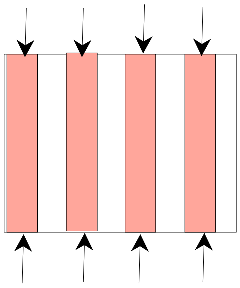

Figure 1: The optimal composite under the homogeneous axial loading.

**Instabilities in the optimal design problems.** The following example demonstrates the instability of the optimal structure and suggests ways of reformulating the problem in order to stabilize the design.

**Problem 1.** Suppose that a square domain $a\,b\,c\,d$ filled with a composite material, is loaded by a uniaxial loading. Suppose for simplicity, that the composite is assembled from the material with unit compliance tensor $\boldsymbol{S}_1 = \boldsymbol{I}$ (the Poisson ratio is equal to zero and the Young modulus is equal to one) and from the void with infinite compliance: $\boldsymbol{S}_2 = \infty$. Suppose also, that the fractions $m_1$ of the material and $m_2$ of the void are equal to one half each:

$$m_1 = m_2 = \frac{1}{2}. \tag{5}$$

Let the domain be loaded by a prescribed loading

$$\boldsymbol{f}_0 = \begin{cases} \boldsymbol{i}_1 & \text{on} \quad a\,b \\ 0 & \text{on} \quad b\,c \\ -\boldsymbol{i}_1 & \text{on} \quad c\,d \\ 0 & \text{on} \quad a\,d \end{cases}. \tag{6}$$

The optimal design is homogeneous. The loading $\boldsymbol{f}_0$ creates a stress field $\boldsymbol{\sigma}_1$,

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad \boldsymbol{\sigma}_1 = \boldsymbol{i}_1 \otimes \boldsymbol{i}_1 \tag{7}$$

inside the domain.

The problem is to find the composite that minimizes the energy of the project under the loading $\boldsymbol{f}_0$. Obviously, the best structure is a simple laminate, with layers oriented along the loading (see figure 1).

The effective compliance $s_{1111}$ in the direction $\boldsymbol{i}_1$ of the loading is equal to the harmonic mean of the (unit) material's complaince $s_m = 1$ and the (infinite) complaince of the void

$s_v = \infty$:

$$s_{1111} = \left(\frac{m_1}{s_m} + \frac{m_2}{s_v}\right)^{-1} = 2 \tag{8}$$

The minimal energy and the problem cost are:

$$W(\boldsymbol{\sigma}_1) = \boldsymbol{\sigma}_1 : \boldsymbol{S}_* : \boldsymbol{\sigma}_1 = s_{1111}\boldsymbol{\sigma}_{11}^2 = 2, \quad H(\boldsymbol{\sigma}_1) = 2 \tag{9}$$

This solution, however, is not satisfactory from a common sense viewpoint. Indeed, the laminate structure is extremely unstable, and its compliance tensor is singular. The laminate structure cannot resist any loading but the prescribed one. Its compliance is infinitely large for all other loadings. Simply speaking, the structure falls apart under any infinitesimally small applied stress that has either shear component, or a component along the axis $\boldsymbol{i}_2$.

**Remark 2.1** *The described instability is typical for the projects that are designed to optimally resist to a prescribed loading, at the expense of the resistivity in other directions.*

**Formulation of the problem of stable optimal design.** Let us consider a problem of energy optimization of an elastic body $\Omega$ loaded by unknown forces $\boldsymbol{f}$ applied on the boundary $\partial\Omega$. In this paper, we focus on the dependence of the optimal project on the loading that belongs to a set $\mathcal{F}$: $\boldsymbol{f} \in \mathcal{F}$. We define the compliance $\Lambda$ of a construction as the maximum of compliances upon all admissible loadings

$$\Lambda(\boldsymbol{p}) = \max_{\boldsymbol{f} \in \mathcal{F}} H(\boldsymbol{p}, \boldsymbol{f}), \tag{10}$$

and we formulate the problem of the optimal design against the "worst" loading:

$$\min_{\boldsymbol{p} \in \mathcal{P}} \Lambda = \min_{\boldsymbol{p} \in \mathcal{P}} \left\{ \max_{\boldsymbol{f} \in \mathcal{F}} H(\boldsymbol{p}, \boldsymbol{f}) \right\}. \tag{11}$$

The energy $H$ is a quadratic functional of $||\boldsymbol{f}||$. To impose constraints on acting forces, we formulate a problem for a design that offers a minimal compliance in a class of loadings.

**Integral constraints for the loading.** Let the set of the loadings $\mathcal{F}$ be characterized by an integral constraint. It is convenient to consider the constraints as a quadratic form of the loading: this form leads to rather simple equations and possesses a needed generality and flexibility. Suppose that an unknown loading by normal forces $\boldsymbol{f} \in \mathcal{F}$ is constrained as following:

$$\mathcal{F} = \left\{ \boldsymbol{f} : \quad \oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi(S) \boldsymbol{f} \leq 1 \right\}, \tag{12}$$

where $\Psi(S)$ is a positively defined weight matrix,

$$\Psi(S) > 0, \quad \forall\ S \in \partial\Omega. \tag{13}$$

The introduced here weight function $\Psi$ expresses a priori assumptions about the unknown loading. For instance, the case when all loadings are equally possible, corresponds to $\Psi = const(S)$.

The compliance of the design, introduced in (10), is given by the solution of the problem of maximization of the stored in the design energy with respect to the applied loadings $\boldsymbol{f} \in \mathcal{F}$:

$$\Lambda = \max_{\boldsymbol{f}} H(\boldsymbol{p}, \boldsymbol{f}) = \max_{\boldsymbol{f}} \min_{\boldsymbol{\sigma}} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi(S) \boldsymbol{f}}. \tag{14}$$

The problem (14) an eigenvalue problem. The value $\Lambda(\boldsymbol{p})$ corresponds to the first eigenfunction or to the set of the eigenfunctions, that generate the most "dangerous" loading(s) from the considered class. Hence we formulate the stable optimal design problem as a problem of eigenvalue optimization:

$$J_* = \min_{\boldsymbol{p} \in \mathcal{P}} \Lambda(\boldsymbol{p}) = \min_{\boldsymbol{p} \in \mathcal{P}} \max_{\boldsymbol{f} \in \mathcal{F}} \min_{\boldsymbol{\sigma} \in \Sigma} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi \boldsymbol{f}}. \tag{15}$$

# 3 Eigenvalue problem

**Saddle point case.** The question of whether or not the multiple eigenvalue case is taking place, depends on the power of the control. It the control $\boldsymbol{p}$ is "weak", that is if the control cannot change the sequence of eigenvalues, then we are dealing with a saddle point situation. In this case, the minimal upon the control $\boldsymbol{p}$ eigenvalue corresponds to a unique eigenfunction $\boldsymbol{f}(\boldsymbol{p})$. The example below illustrates this situation.

In this case the functional $\Lambda$ (14) is a saddle function of the arguments, and the operations of max with respect to $\boldsymbol{f}$ and min with respect to $\boldsymbol{\sigma}$ can be switched. Then varying the functional, we find the Euler equations for the most dangerous loading. Let us find this loading. Variation of (14) with respect to $\boldsymbol{f}$ gives:

$$\delta \Lambda_{\boldsymbol{f}} = -\left( \oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi(S) \boldsymbol{f} \right)^{-1} (\boldsymbol{u} - \Lambda \Psi \boldsymbol{f})\, \delta \boldsymbol{f}, \tag{16}$$

which implies the relation pointwise between the optimal loading and the boundary deflection

$$\boldsymbol{f}(S) = \frac{1}{\Lambda} \Psi^{-1} \boldsymbol{u}(S), \quad \forall S \in \partial\Omega. \tag{17}$$

It is also easy to see that the stationary condition corresponds to the *maximum* not the minimum of the functional using the second variation technique.

The problem of the most dangerous loading $\boldsymbol{f}_0$ becomes an eigenvalue problem

$$\frac{1}{\Lambda} = \min_{\boldsymbol{\sigma}} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{u} \cdot \Psi(S)^{-1} \boldsymbol{u}}. \tag{18}$$

The cost $\Lambda$ corresponds to the minimal eigenvalue, given by the Rayleigh ratio (18), and the most "dangerous" loading corresponds to the first eigenfunction of this problem.

**Remark 3.1** *One can consider also the problem of the most "favorable" loading, that is*

$$\Lambda_- = \min_{\boldsymbol{f}} \min_{\boldsymbol{\sigma}} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} \boldsymbol{f} \cdot \Psi \boldsymbol{f}}. \tag{19}$$

*However, $\Lambda_-$ is zero. Clearly, the spectrum of the operator is clustered at zero. A minimizing sequence is formed from often oscillating forces.*

**Euler equations.** The Euler equations (with respect to $\boldsymbol{\sigma}$) are

$$\begin{array}{ll} \nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma} = \boldsymbol{S}^{-1}(\boldsymbol{p}) : (\nabla \boldsymbol{u})^s & \text{in } \Omega, \\ \boldsymbol{u} = \Lambda \Psi \boldsymbol{\sigma} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{array} \tag{20}$$

They describe the vibration of the body with inertial elements concentrated on $\partial\Omega$.

The problem admits the following physical interpretation: the optimal loading forces are equal to a distribution of inertial elements (concentrated masses) on the boundary component $\partial\Omega$. The specific inertia is described by the tensor $\Psi$, so it could include the resistance to the turning as well. The vibration of such loaded system excites the forces that are proportional to the deflection $\boldsymbol{u}$. The compliance is proportional to the eigenfrequency of vibrations. One can see that the introduced quantity $\Lambda$ characterizes the domain or the construction itself, it represents the maximum of possible stored energy under any loading from the class $\mathcal{F}$.

These equations form an eigenvalue problem that possesses infinitely many solutions. We pick up the pair $\{\Lambda_1, \ \boldsymbol{\sigma}_1\}$ that corresponds to the maximal eigenvalue $\Lambda_1 = \max\{\Lambda_k\}$.

The problem (18) with unit matrix $\Psi$ is called the Steklov eigenvalue problem, which considers the ratio of integrals of different dimensionality. The corresponding Euler equation (20) has an eigenvalue in the boundary condition. Similar optimality conditions were derived in [5] for the optimal boundary sources in electrical tomography problem.

**Problem 2. Stable optimal design of a loaded beam.** The problems for beams and bending plates admit the loading distributed in the whole domain of the definition: on the interval in the case of the beam, and in the plane domain in the case of the bending plate or shell. In these problems, the loaded surface $\partial\Omega$ coincide with the domain $\Omega$ itself.

Consider an elastic beam whose energy density is

$$W = p(w'')^2 - 2fw, \tag{21}$$

where $p \geq 0$ is a material's stiffness, that can be varied from point to point and subject to the integral constraint

$$\int_0^l p \, dx = V \tag{22}$$

which expresses the limits on resources; $f$ is the intensity of the normal loading, subject to the constraint

$$\int_0^l f^2 \, dx = 1. \tag{23}$$

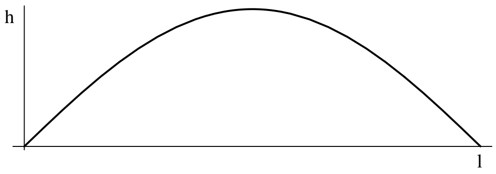


Figure 2: The stiffness $p(x)$ of the optimal beam.

Consider an optimization problem of choosing a stiffness $p(x)$ that maximally resists to the most dangerous loading $f$:

$$\min_{p\geq 0, p\in(22)} \max_{f} \min_{w} \mu, \quad \mu = \frac{\int_0^l (p(w'')^2 - 2fw)\, dx}{\int_0^l (f^2)\, dx}. \tag{24}$$

The stationary conditions are:

$$\delta\, w: \quad (pw'')'' - f = 0, \ \forall x \in (0,\ 1), \quad pw''|_{x=0} = pw''|_{x=l} = 0, \tag{25}$$

$$\delta\, f: \quad f + \frac{w}{\mu} = 0, \ \forall x \in (0,\ 1), \tag{26}$$

$$\delta\, p: \quad (w'')^2 = \gamma, \ \forall x \in (0,\ 1), \tag{27}$$

where $\gamma$ is the Lagrange multiplier for the constraint (22). This system admits a solution

$$w = -\mu f = -\gamma x(l-x)/2, \tag{28}$$

$$f = \frac{\sqrt{\gamma}}{\mu} x(l-x)/2, \tag{29}$$

$$p_0 = \frac{1}{\mu}\frac{x}{24}(x-l)\left((x-\frac{l}{2})^2 - \frac{5l}{4}\right). \tag{30}$$

Accounting for the constraints, we get

$$\mu = \frac{l^5}{5!V}, \tag{31}$$

$$p(x) = \frac{5V}{4l^5} x(l-x)\left(5l^2 - (2x-l)^2\right), \tag{32}$$

$$w(x) = -\frac{1}{V}\sqrt{\frac{l^5}{5!}}\frac{x(l-x)}{2}. \tag{33}$$

The optimal stiffness $p(x)$ of the beam is shown on figure 2. Interestingly, that the optimal solution is found analytically.

**A min-max problem for a partly known loading.** The developed formulation can be generalized to the case of a partly known loading. Suppose, that the applied loading $\boldsymbol{f}$

is composed of the known component $\boldsymbol{f}_0$ and of some unknown component $\boldsymbol{f}_1$ which describes an uncertainty in the loading. The constraint has the form

$$\mathcal{F}_1 = \left\{ \boldsymbol{f}_1 : \quad \oint_{\partial\Omega} \boldsymbol{f}_1 \cdot \Psi(S) \boldsymbol{f}_1 \leq 1 \right\}, \tag{34}$$

or

$$\oint_{\partial\Omega} (\boldsymbol{f} - \boldsymbol{f}_0) \cdot \Psi(S)(\boldsymbol{f} - \boldsymbol{f}_0) \leq 1. \tag{35}$$

The problem is no longer homogeneous. The functional is

$$I = \min_{\boldsymbol{p}\in\mathcal{P}} \Lambda(\boldsymbol{p}) = \min_{\boldsymbol{p}\in\mathcal{P}} \max_{\boldsymbol{f}_1\in\mathcal{F}_1} \min_{\boldsymbol{\sigma}\in\Sigma} \frac{\int_\Omega W(\boldsymbol{p}, \boldsymbol{\sigma})}{\oint_{\partial\Omega} (\boldsymbol{f} - \boldsymbol{f}_0) \cdot \Psi(S)(\boldsymbol{f} - \boldsymbol{f}_0)}. \tag{36}$$

Depending on $\mathcal{P}$ and $\mathcal{F}_1$, we can encounter two different situations: minimization of a single or multiple maximal eigenvalues $\Lambda(\boldsymbol{p})$. In the first case, the stationary conditions with respect to $\boldsymbol{f}$ and $\boldsymbol{\sigma}$ lead to the Euler equation:

$$\nabla \cdot \boldsymbol{S}^{-1} (\nabla\, \boldsymbol{u})^s = 0 \quad \text{in } \Omega, \tag{37}$$

$$\boldsymbol{u} - \Lambda\Psi \cdot \boldsymbol{S}^{-1} : (\nabla\, \boldsymbol{u})^s \cdot \boldsymbol{n} = \boldsymbol{f}_0 \quad \text{on } \partial\Omega. \tag{38}$$

They describe the inhomogeneous boundary value problem, which has a unique solution. The optimality condition

$$\frac{\partial W(\boldsymbol{p}, \boldsymbol{\sigma})}{\partial \boldsymbol{p}}\, \delta\, \boldsymbol{p} \geq 0, \tag{39}$$

has different forms depending on the type of used control and of possible isoperimetric restrictions.

Discussed below problems 3 and 4 illustrate the problem of a design with partly known loading. The occurrence of multiple eigenvalues in the problem 3 switches in the problem 4 to a situation when the problem with a single maximal eigenvalue becomes one with multiple eigenvalues, if we change slightly the set of design parameters $\mathcal{P}$ or the set of possible variations in the loading $\mathcal{F}_1$.

## 4 Multiple eigenvalues

**Eigenvalue optimization.** We return to the discussion of the project that minimizes the functional $\Lambda$, or minimizes the stored energy in the most unfortunate situation. The problem has the form (15). The specific effect of the min-max problem is the possibility of appearance of multiple eigenvalues. The mechanism of this phenomenon is the following. Minimization of the maximal eigenvalue likely leads to the situation when its value meets the second eigenvalue of the problem. In this case, both eigenvalues must be minimized together, until their common value reaches the third eigenvalue, and so on. The multiplicity means that two or more loadings give the same value of the problem. We will bring below an example demonstrating this phenomenon: the resistance of the construction to five different loadings

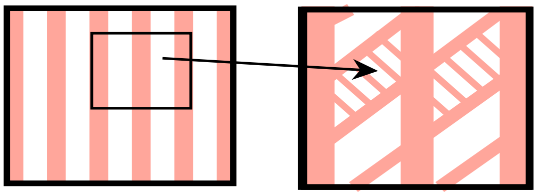

Figure 3: The schematic picture of the composite of the third rank.

in this example of the stable optimal design is the same. The second example of that section illustrates the situation when the eigenvalue could either be multiple or stay single, depending on the set of loadings $\mathcal{F}$ and the class of admitted materials $\mathcal{P}$, constraining controls $\boldsymbol{p} \in \mathcal{P}$. This is a problem with a partly known loading, the Euler equation of which for one most dangerous loading was derived in the previous section. Similar min-max problem with multiple eigenvalues was considered in [4] for nondestructive testing of the worst possible damage by applying optimal boundary currents.

There is an extended literature on eigenvalue optimization. It was understood in a different setting: the maximization of the fundamental frequency. We refer to the recent review papers [6, 11] and references therein.

**Optimal composite structures.** Consider the following problem of structural optimization. A domain made of a two-phase composite material of an arbitrary structure is loaded by an uncertain loading $f_0$. We want to find the most resistant structure of the composite, that is to minimize the functional (10). Here $\boldsymbol{p}$ is a vector of parameters that defines the tensor $\boldsymbol{S}_*$ of the effective compliance of the composite. For definiteness, consider the two-dimensional elasticity problem.

We do not know a priori, how many loadings should be taken in consideration. But clearly, it is sufficient to enlarge the set of admissible composites to those which minimize the sum of elastic energies caused by any number of different loadings. These composites are described in the paper by Avellaneda [1]: in two-dimensional elasticity, they form the class of the so-called matrix laminates of the third rank (see figure 3).

The effective property tensors of these composites admit an analytical expression through their structural parameters. To describe the class of the effective tensors of these anisotropic structures, we use the natural tensor basis

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{40}$$

Any stress and strain matrices are represented as vectors in their basis, and the effective compliance $\boldsymbol{S}_*$ of matrix laminates is given by the $3 \times 3$ matrix (see [8])

$$\boldsymbol{S}_* = \boldsymbol{S}_1 + m_2 \left( (\boldsymbol{S}_2 - \boldsymbol{S}_1)^{-1} + \frac{2m_1}{E_1} \boldsymbol{N} \right)^{-1}, \tag{41}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are the compliance matrices of the first and the second materials, $m_1$ and $m_2$ are the volume fractions, $E_1$ is the Young modulus of the first material (which forms the envelope). The matrix $\boldsymbol{N}$ depends on the structural parameters: on the angles $\theta_i$ between the tangent to the laminates and the axis $\boldsymbol{i}_1$, and on the relative thickness $\alpha_i$ (see figure 3):

$$\boldsymbol{N} = \sum_i^3 \alpha_i \begin{pmatrix} \cos^4\theta_i & \cos^2\theta_i \sin^2\theta_i & \cos^3\theta_i \sin\theta_i \\ \cos^2\theta_i \sin^2\theta_i & \sin^4\theta_i & \cos\theta_i \sin^3\theta_i \\ \cos^3\theta_i \sin\theta_i & \cos\theta_i \sin^3\theta_i & \cos^2\theta_i \sin^2\theta_i \end{pmatrix}, \quad \sum_i^3 \alpha_i = 1, \quad \alpha_i \ge 0. \tag{42}$$

$\alpha_i$ and $\theta_i$ form the control vector $\boldsymbol{p}$.

**Problem 1 revisited. Unstable design for a uniaxial loading.** Discussing the instabilities of the optimal project in the problem 1 below, we considered the optimization problem

$$\min_{\boldsymbol{p}\in\mathcal{P}} H(\boldsymbol{p}, \boldsymbol{f}_0), \tag{43}$$

where $\boldsymbol{f}_0$ is given by (6), and the set $\mathcal{P}$ constrains the parameters of the composite $\alpha_i$ and $\theta_i$. The solution of the problem is a laminate, that is easily found from (41), (42). It corresponds to the parameters $\alpha_1 = 1,\ \theta_1 = 0,\ \alpha_2 = \alpha_3 = 0$. This structure is shown on figure 1, below we discussed the instabilities of this solution. Indeed, the compliance tensor $\boldsymbol{S}_*$ of a third rank composite becomes $(m_1 = m_2 = 1/2)$

$$\boldsymbol{S}_* = \boldsymbol{S}_1 + \frac{1}{2}(\boldsymbol{N})^{-1}. \tag{44}$$

For the optimal choice of the parameters $\alpha_i,\ \theta_i$ the matrix $\boldsymbol{N}$ (see (42)) has two zero eigenvalues, and the two eigenvalues of $\boldsymbol{S}_*$ corresponding to the shear loading and the loading in the direction $\boldsymbol{i}_2$, are infinite (see (41)). Hence the compliance of the structure is infinitely large for any loading that has a projection on these two eigenvectors.

**Problem 3. Stable design for a uniaxial loading.** Now we reformulate the design problem (43) to obtain a stable project. Suppose that the loading is not exactly known. Namely, the loading field $\boldsymbol{\sigma}$ can take one of the following six values $\boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i, \quad i = 1, ...6$, where $\boldsymbol{\sigma}_1$ is given by (7) and

$$\boldsymbol{\tau}_{1,2} = \pm r\boldsymbol{e}_1 = \begin{pmatrix} \pm r & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\tau}_{3,4} = \pm r\boldsymbol{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \pm r \end{pmatrix},$$
$$\boldsymbol{\tau}_{5,6} = \pm r\boldsymbol{e}_3 = \begin{pmatrix} 0 & \pm r/\sqrt{2} \\ \pm r/\sqrt{2} & 0 \end{pmatrix}. \tag{45}$$

Here, $r > 0$ is a real parameter. The additional loadings of the magnitude $r$ corresponding to the cases 1, 3, 5 are shown on figure 4. The 'twin' loadings correspond to the reverse directions of the forces.

Assume, in addition, that $r$ is smaller than the magnitude of the "main" loading, which in our example is equal to one. The six loadings are viewed as small perturbations of the

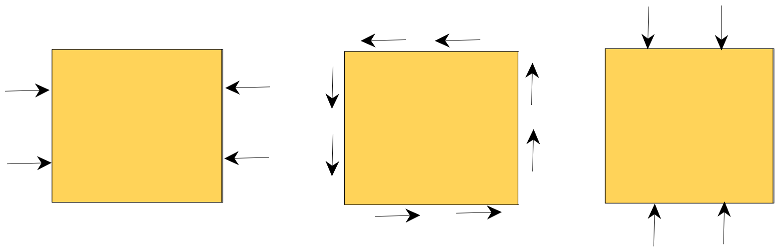

Figure 4: The additional loadings of the magnitude $r$.

main loading, that correspond to all linearly independent directions of the symmetric tensor $\boldsymbol{\sigma}$. In spite of the smallness of $r$, the perturbation of the functional (43) is infinitely large, if $\boldsymbol{S}_*$ is optimally chosen. This characterizes the instability of the optimal project to those perturbations.

Let us reformulate the optimization problem. We are looking for a structure of a composite that minimizes the *maximum* of compliances $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)$ upon all considered loadings.

$$\min_{\boldsymbol{p}=\{\alpha_i,\ \theta_i\}} \left\{ \max_{i=1..6} H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i) \right\}. \tag{46}$$

The obtained min-max problem asks for the minimal compliance in the case of the "most dangerous" loading. To construct the solution of the optimization problem, we introduce a variable $z$ that is greater than any of $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)$,

$$z \geq H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i), \quad i = 1, ...6. \tag{47}$$

The problem (46) can be formulated as follows (see [7]):

$$\min_{\boldsymbol{p}} \left\{ z + \sum_{i}^{6} \lambda_i^2 (z - H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)) \right\}, \tag{48}$$

where $\lambda_i^2$ are the non-negative Lagrange multipliers by the constraints (47). The Lagrange multiplier is equal to zero, if this relation is satisfied as a strong inequality, and is non-zero, if it is satisfied as an equality [7]:

$$\lambda_i^2 = \begin{cases} = 0 & \text{if} \quad z > H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i) \\ > 0 & \text{if} \quad z = H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i) \end{cases}. \tag{49}$$

The problem requires minimization of the weighted sum of energies of the 'dangerous' loadings $(\boldsymbol{\tau}_i)$, $i \in I$ . Here $I$ is the set of such 'dangerous' loadings. Other loadings lead to the smaller energies $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_j)$: $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_j) < H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i)$, if $i \in I,\ j \notin I$, and therefore to $\lambda_j = 0$. This leads to the equalities

$$\begin{array}{l} z = H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i), \quad \text{if} \ \ i \in I, \\ z > H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_i), \quad \text{if} \ \ i \notin I \end{array}. \tag{50}$$

Applying to the problem (48), we argue that the set of dangerous loadings in this example consists of five elements, $I = \{1, 3, 4, 5, 6\}$ :

$$H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_1) > H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_2), \tag{51}$$

$$\begin{aligned} H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_3) &= H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_4), \\ H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_5) &= H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_6). \end{aligned} \tag{52}$$

The inequality (51) is explained by the observation that an additional loading, if codirected with the main load, will either increase or decrease its magnitude independently of the composite structure. Clearly, the energy of the more intensive loading is greater.

The symmetry of the loadings # 3 and #4 and of the loadings # 5 and # 6 together with the symmetry of the set of admissible structural tensors $\mathcal{P}$ suggests that the "twin" loadings lead to the same cost of the problem. In other words, the same project $\boldsymbol{p}$ minimizes both $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_3)$ and $H(\boldsymbol{p}, \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_4)$, keeping them equal to each other; the same for the other pair of loadings. To achieve the equalities (52), we require the symmetry of the would be optimal tensor $\boldsymbol{S}_*$ (see (41)):

$$\alpha_1 = 1 - a, \quad \alpha_2 = \alpha_3 = a/2, \quad \theta_1 = 0, \quad \theta_2 = -\theta_3 = \theta, \tag{53}$$

where $a$ and $\theta$ are two parameters. Physically, we require the orthotropy of $\boldsymbol{S}_*$.

Under the conditions (53), the matrix $\boldsymbol{N}$ (see (42)) takes the form

$$\boldsymbol{N} = (1-a)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a\begin{pmatrix} \cos^4\theta & \cos^2\theta\sin^2\theta & 0 \\ \cos^2\theta\sin^2\theta & \sin^4\theta & 0 \\ 0 & 0 & \cos^2\theta\sin^2\theta \end{pmatrix} \tag{54}$$

and, from (44)

$$\begin{aligned} \boldsymbol{S}_* &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2a(1-a)}\begin{pmatrix} a & -a\cot\theta & 0 \\ -a\cot\theta & T & 0 \\ 0 & 0 & (1-a)\csc^2\theta\sec^2\theta \end{pmatrix}, \\ T &= \frac{1}{8}(8 - 5\,a + 4\,a\,\cos 2\,\theta + a\,\cos 4\,\theta)\,\csc^4\theta. \end{aligned} \tag{55}$$

**Remark 4.1** *Note, that the matrix becomes singular when $a \to 0$, which corresponds to unstable design.*

**Optimal parameters.** The described class of symmetric composites is defined by two parameters $\theta$ and $a$. The symmetry of the project eliminates the necessity to compare the loadings except from those with numbers 1, 3, 5. It turns out that these loadings are equally "dangerous":

$$H(a, \theta; \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_1) = H(a, \theta; \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_3) = H(a, \theta; \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_5). \tag{56}$$

Two equalities (56) allow to compute the optimal values of $\theta$ and $a$. One can easily see that the problem is always solvable. The optimal values of the parameters $\theta$ and $a$ correspond to the solution of the min-max problem:

$$J(a, \theta) = \min_{a,\theta}\left\{\max\left\{H(a, \theta; \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_1),\ H(a, \theta; \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_3),\ H(a, \theta; \boldsymbol{\sigma}_1 + \boldsymbol{\tau}_5)\right\}\right\}. \tag{57}$$

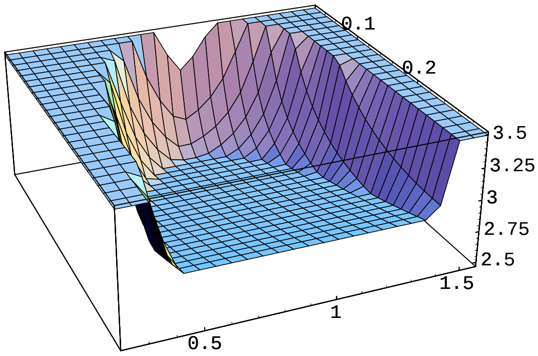


Figure 5: The graph of the function $J(a, \theta)$.

Note, that the project (55) is not optimal for any single loading but it is optimal for the set of them. The solution provides an example of a mixed strategy in the game: loadings versus design.

**Illustration.** Set $r = 0.1$. The graph of the function $J(a, \theta)$ is presented on figure 5. The magnified neighborhood of the minimum is shown on figure 6, where the viewpoint is changed for better visibility. The optimal values of the parameters are $\theta = 0.889$, $a = 0.0496$, $J = 2.483$. We see that the compliance is bigger than the compliance of the construction optimal for a single load. On the other hand, the found construction is stable to all loadings, unlike the original design.

The picture of the optimal structure is shown on the figure 3. Note that a part of the material is removed from the laminates that resist the main load. This material is placed in "reinforcements" that reduce the compliance in all directions.

**The sensitivity.** The non-smoothness of the min-max formulation leads to the nonlinear 'sensitivity' to the perturbations. Particularly, the optimal design may stay the same independently of sufficiently small perturbations of the loading. To demonstrate this, consider the following problem.

**Problem 4**. Consider the previous optimization problem, but suppose that both materials in the mixture have finite stiffness. The structure for this problem belongs to the same class of the third-rank laminates. The scheme of the solution is quite similar to the discussed case. However, there is an important difference. The maximal compliance in any direction is no longer an infinity, but it is estimated (see [8]) by the arithmetic mean of the compliances of the materials. Assume that the compliance matrices are equal to $s_1 \boldsymbol{I}$ and $s_2 \boldsymbol{I}$, and the volume

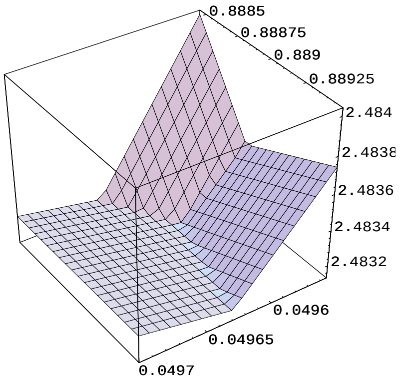


Figure 6: The magnified neighborhood of the minimum of the functional.

fractions are equal to one half. The compliance of the laminate material is

$$\boldsymbol{S}_{lam} = \begin{pmatrix} \frac{2s_1s_2}{s_1+s_2} & 0 & 0 \\ 0 & \frac{s_1+s_2}{2} & 0 \\ 0 & 0 & \frac{s_1+s_2}{2} \end{pmatrix}. \tag{58}$$

If the perturbation parameter $r$ is small enough, then the maximum of compliance can correspond to one loading only.

Indeed, the energy of the loading $\boldsymbol{\sigma}_1$ in the initial laminate structure $\boldsymbol{p}_1$ is equal to

$$W_1(\boldsymbol{p}_1) = (1+r)^2 \frac{2s_1s_2}{s_1+s_2}. \tag{59}$$

For each of the loadings $\boldsymbol{\tau}_i,\ i=3,...,6$ there is a structure $\boldsymbol{p}_i$ which minimizes its energy. The energy $W_i(\boldsymbol{p}_1)$ of the loadings $\boldsymbol{\tau}_i,\ i=3,...,6$ in the initial laminated structure is clearly greater than the energy $W_i(\boldsymbol{p}_i)$ corresponding to the optimal structure $\boldsymbol{p}_i$. Therefore $W_i(\boldsymbol{p}_i)$ is bounded as

$$W_i(\boldsymbol{p}_i) \le W_i(\boldsymbol{p}_1) = \frac{2s_1s_2}{s_1+s_2} + r^2\frac{s_1+s_2}{2}, \quad i=3,...,6. \tag{60}$$

If the upper bound of $W_i$ is still smaller than $W_1$: $W_i(\boldsymbol{p}_1) < W_1(\boldsymbol{p}_1),\quad i=3,...,6$, then the laminate is optimal, because $W_1$ reaches its minimum on it. The corresponding condition for $r$ is

$$0 \le r < \frac{8s_1s_2}{(s_1-s_2)^2}. \tag{61}$$

If $r$ satisfies this inequality, then the laminate structure remains optimal. This means that the design is insensitive to sufficiently small but final perturbations of the loading. If (61) holds, then the set of dangerous loadings consists of the first loading only.

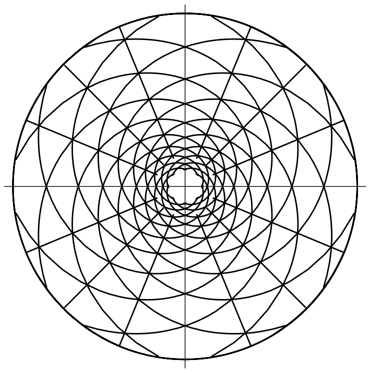

Figure 7: The cartoon of the optimal structure of the wheel.

**Symmetry. The optimal wheel.** The next example of the multiplicity of the "most dangerous" loadings demonstrates the appearance of symmetric projects in a min-max optimal design problem.

**Problem 5.** Consider the problem of a design of an optimal wheel. A circular domain is loaded by a non-axisymmetric loading that consists of a pair of radial forces applied to the rim and to the hub. These forces can move circumferential, which corresponds to the revolution of the wheel. If a loading $\boldsymbol{f}(S)$ is admissible, then any shifted loading $\boldsymbol{f}(S+\theta)$ is admissible too. Here, $S$ is the circumferential coordinate and $\theta$ is an arbitrary real number.

Consider an optimal design problem. Suppose, that it is required to minimize the maximal compliance of the wheel in a class of forces. The design which minimizes the maximal compliance is obviously axisymmetric even if a particular loading is not. The symmetry comes from the min-max requirement of the equal resistance to all forces $\boldsymbol{f}(S+\theta)$: the project is independent of the angle $\theta$.

The optimal axisymmetric layout of the composite properties $\boldsymbol{S}_*(\rho)$ in any particular point $\rho$ minimizes the integral over $\theta$ of the energy distribution. The solution locally is again the third rank laminate, symmetric with respect to angular coordinate $\theta$. The properties of the structure vary with the radius.

In the large, it can be represented as a periodic system of radii and two symmetric spirals (see figure 7). The period of the spirals is infinitesimal, and the thickness of the materials varies with radius.

# 5 Acknowledgements

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