MATH 6720
Final
Due Date: 05/03/2017 12pm

Instructions:

- There is a two hours time limit to complete the midterm. The time limit is enforced on the honor system, do not spend the entire weekend thinking about the problems.
- Verbal or electronic collaborations are not allowed.
- Notes, books, electronic material are not allowed.

1. **Conformal mappings**
   (a) Show that the transformation $\xi = 1/z$ maps the line $x = c_1 \neq 0$ to a circle with center along the real axis.
   (b) A Möbius transformation maps the region between the non-concentric circles $|z| = 1$ and $|z - 13/4| = (15/4)^2$ onto an annulus $\rho_0 < |z| < 1$. Find ρ_0 only, i.e you don't need to give the transformation.

2. **Green's function**
   Find the Green's function for the operator $(L - \lambda)u = \delta(x - \xi)$, $\lambda \neq 0$, $Lu = -u''$ on $[0, 1]$ with boundary conditions $u'(0) = u(1) = 0$.
   HINT: $\sin(u)\sin(v) = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$ and $\cos(u)\cos(v) = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$.

3. **Asymptotic expansion of integrals**
   Find the leading order behavior and show that the relationship is asymptotic.
   (a)
   $$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt \quad x \to 0^+.$$
   (b)
   $$I(x) = \int_x^\infty e^{-t^3} dt \quad x \to +\infty.$$

4. **Watson's lemma**
   Show that the complete asymptotic expansion of
   $$I(x) = \int_0^\infty (t^2 + 2t)^{-1/2} e^{-xt} dt$$
   is
   $$I(x) \sim \sum_{n=0}^\infty (-1)^n \frac{[\Gamma(n + 1/2)]^2}{2^{n+1/2} n! \Gamma(1/2) x^{n+1/2}} \quad x \to +\infty.$$
   HINT: The Gamma function satisfies $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$.