MATH 6750
Homework # 3
Due Date: November 20, 2019 at the beginning of class.

1. Consider the velocity field for a Newtonian fluid between two infinite parallel disks that are separated by a gap H. The lower disk rotates about the central axis with angular velocity Ω. Assume that the other disk is stationary. The governing equations are given below.

   Continuity:
$$\frac{1}{r}\left[\partial_r(ru_r) + \partial_\theta u_\theta\right] + \partial_z u_z = 0.$$

   r-momentum:
$$\rho\left(u_r\partial_r u_r + \frac{u_\theta}{r}\partial_\theta u_r + u_z\partial_z u_r - \frac{u_\theta^2}{r}\right) = -\partial_r p + \mu\left[\partial_r\left(\frac{1}{r}\partial_r(ru_r)\right) + \frac{1}{r^2}\partial_{\theta\theta}u_r - \frac{2}{r^2}\partial_\theta u_\theta + \partial_{zz}u_r\right]$$

   θ-momentum:
$$\rho\left(\partial_t u_\theta + u_r\partial_r u_\theta + \frac{u_\theta}{r}\partial_\theta u_\theta + u_z\partial_z u_\theta + \frac{u_\theta u_r}{r}\right) = -\frac{1}{r}\partial_\theta p + \mu\left[\partial_r\left(\frac{1}{r}\partial_r(ru_\theta)\right) + \frac{1}{r^2}\partial_{\theta\theta}u_\theta + \frac{2}{r^2}\partial_\theta u_r + \partial_{zz}u_\theta\right]$$

   z-momentum:
$$\rho\left(\partial_t u_z + u_r\partial_r u_z + \frac{u_\theta}{r}\partial_\theta u_z + u_z\partial_z u_z\right) = -\partial_z p + \mu\left[\partial_r\left(\frac{1}{r}\partial_r(ru_z)\right) + \frac{1}{r^2}\partial_{\theta\theta}u_z + \partial_{zz}u_z\right]$$

   The boundary conditions are:
   At $z = 0$, $u_r = u_z = 0$ and $u_\theta = \Omega r$.
   At $z = H$, $u_r = u_z = u_\theta = 0$.

   1. Nondimensionalize the equations. Note that there are two choices for the characteristic pressure either $p_c = \Omega\mu$ or $p_c = \rho\Omega^2 H^2$. Explain how the second choice is the correct one when considering the limit $\mathrm{Re} \to 0$, while the first one is to be used in the limit $\mathrm{Re} \to 0$.
   2. Solve the equation in the creeping flow limit, i.e $\mathrm{Re} = 0$. Note that the solution is radially symmetric and independent of θ.
   3. Use a regular perturbation series to obtain the first order solution, i.e. up to and including terms $O(\mathrm{Re})$.
   4. Use Matlab or another software to plot the secondary flow (the first order perturbation).

2. Consider a general similarity solution $\Psi = F(x)f(\eta)$ with $\eta = y/g(x)$ to the boundary layer equations
$$u\partial_x u + v\partial_y u = -\frac{1}{\rho}\partial_x p + \nu\partial_{yy}u \quad \partial_x u + \partial_y v = 0$$
   where Ψ is a the stream function.

   1. Show that the condition, $u \to U(x)$ as $y/\delta \to \infty$, demands that $F(x) = cU(x)g(x)$, where c is a constant, and then choose c to be 1.

2. Show that substitution in the boundary layer equations leads to

$$f'^2 - \left(1 + \frac{U}{U'}\frac{g'}{g}\right) f f'' = 1 + \frac{\nu f'''}{g^2 U'}.$$

Eliminate the pressure by matching to the outer solution (Bernoulli), i.e.

$$-\frac{1}{\rho}\partial_x p = u_e \partial_x u_e = U(x)U'(x)$$

3. Deduce that a similarity solution is only possible if either $U(x) \sim (x - x_0)^m$ or $U(x) \sim e^{\alpha x}$, where x_0, m, α are constants.
4. In the case, $U(x) = Ax^m$, $A > 0$, show that $g(x)$ is proportional to $x^{(1-m)/2}$, and show that choosing

$$g(x) = \sqrt{\frac{2\nu}{(m+1)Ax^{m-1}}}$$

leads to

$$f''' + f f'' + \frac{2m}{m+1}(1 - f'^2) = 0,$$

subject to $f(0) = f'(0) = 0$ and $f'(\infty) = 1$.

3. Consider uniform slow flow past a circular cylinder, and show that the problem reduces to

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \Psi = 0,$$

with $\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \theta} = 0$ on $r = a$ and $\Psi \sim Ur\sin\theta$ as $r \to \infty$. Show that seeking a solution of the form $\Psi = f(r)\sin\theta$ leads to

$$\Psi = \left[Ar^3 + Br\log r + Cr + \frac{D}{r}\right]\sin\theta$$

and thus fails, in that for no choice of the arbitrary constants can all the boundary conditions be satisfied.