An Introduction to the theory of $(\mathfrak{g}, K)$-modules

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Introduction

Let G be a reductive Lie group (e.g. $GL(2, \mathbb{R})$) and $\mathfrak{g}_0$ its Lie algebra. One of the longstanding questions in representation theory is to describe $\widehat{G}$ the space of all irreducible representations of G up to isomorphism. If G was compact, there is a complete classification as a result of the Peter-Weyl and Borel-Weil-Bott theorems. Therefore, we shall consider when G to be non-compact. The problem in this case is that, unlike for compact groups, almost all of the irreducible representation of G are infinite dimensional. These representations are significantly harder to study.

In [HC53] Harsih-Chandra developed a systematic way of transforming infinite dimensional representations of G into a purely algebraic object. In the broadest scope, he associated to each representation (π, H) of G a subspace H_{fin} which comes equipped with the structure of a representation of the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and of a maximal compact subgroup K that are compatible (see §2.4 for more details). These new objects, aptly called $(\mathfrak{g}, K)$-modules, form a suitably nice category $\mathscr{C}(\mathfrak{g}, K)$. The natural functor

$$\mathbf{Mod}(G) \to \mathscr{C}(\mathfrak{g}, K)$$

sends irreducible modules to irreducible modules and thus reduces the problem of finding all irreducible representations of G to finding those simple $(\mathfrak{g}, K)$-modules. There turn out to be many important classes of these modules, the largest one being the *admissible* modules (aka. Harish-Chandra modules). These are, in some sense, the class of well behaved $(\mathfrak{g}, K)$-modules. In [Lan89] [1] Langlands discovered a parametrization of the *admissible dual* $\widehat{G}_{adm}$. The goal of this text is to provide a thorough introduction to the infinite dimensional representation theory of real reductive groups and will culminate with the Langlands classification of admissible modules as well as the Knapp-Zuckerman classification of tempered modules. Once this is done, we shall conclude the text with an exposition on the Vogan-Zuckerman classfication of unitary representations with certain non-zero $(\mathfrak{g}, K)$-cohomology.

[1] Langlands originally published this result in 1973.

Chapter 1

Structure Theory of Real Lie Groups

Before we get to any representation theory, we shall first discuss some key decompositions of reductive Lie groups and Lie algebras. As it will turn out, using these decompositions we will get a complete picture of the topology of the Lie groups as well as some interesting information about integration on these manifolds.

1.1 Cartan Involutions

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. It is well known that $\mathfrak{g}$ admits a non-degenerate symmetric bilinear form $B(-,-)$ (the Killing form) which is invariant under the Lie bracket: namely $B([X,Y],Z) = B(X,[Y,Z])$.

Definition 1.1.1. A **real form** of $\mathfrak{g}$ is a real Lie subalgebra $\mathfrak{g}_0$ such that $\mathfrak{g} \cong (\mathfrak{g}_0)_{\mathbb{C}} \cong \mathfrak{g}_0 + i\mathfrak{g}_0$. More generally, a real form for a finite-dimensional complex vector space V is a pair (V_0, J) consisting of a real subspace V_0 and an $\mathbb{R}$-linear endomorphism J of V such that $J^2 = -\operatorname{Id}_V$ and $V = V_0 \oplus JV_0$. In this case the complex multiplication is given by $(a+bi)v = av + bJ(v)$.

Theorem 1.1.2. *Every semisimple Lie algebra over $\mathbb{C}$ admits a real form. Furthermore, this real form is compact.*

Lemma 1.1.3.

(a) Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then $\mathfrak{g}$ is compact if and only if the Killing form is strictly negative definite.

(b) Every compact Lie algebra can be written as a direct sum $\mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$ where $\mathfrak{z}(\mathfrak{g})$ denotes the center and $[\mathfrak{g},\mathfrak{g}]$ is semisimple and compact.

Proof. We shall prove both statements simultaneously, noting that $(b) \implies (a)$ once we show that the condition on the Killing form is sufficient. Suppose $\mathfrak{g}$ is a real Lie algebra whose Killing form is strictly negative definite. Let $O(B)$ denote the group of linear transformations of $\mathfrak{g}$ fixing B. Then $O(B)$ is a compact subgroup of $GL(\mathfrak{g})$. As $\mathfrak{g}$ is semisimple, $\operatorname{Int}(\mathfrak{g})$ is closed in $\operatorname{Aut}(\mathfrak{g}) \subseteq GL(\mathfrak{g})$. Hence, $\operatorname{Int}(\mathfrak{g})$ is compact and $\mathfrak{g}$ is compact.

Now let $\mathfrak{g}$ be an arbitrary compact Lie algebra. There exists a strictly positive definite quadratic form on $\mathfrak{g}$ such that for a given basis $X_1, ..., X_n$, Q is given by

$$Q(\sum a_i X_i) = \sum a_i^2$$

Furthermore, this form is invariant under $\mathrm{Int}(\mathfrak{g})$. Each inner automorphism of $\mathfrak{g}$ is represented by an orthogonal matrix and each ad X is represented by a skew-symmetric matrix (a_{ij}). As the center of $\mathfrak{g}$ is invariant under $\mathrm{Int}(\mathfrak{g})$ consider the orthogonal complement of $\mathfrak{z}(\mathfrak{g})$ with respect to Q. Call this $\mathfrak{g}'$. As this subalgebra is necessarily invariant under $\mathrm{ad}(\mathfrak{g})$, we know that $\mathfrak{g}'$ is an ideal. Therefore, the Killing form of $\mathfrak{g}'$ is the restriction of the Killing form on $\mathfrak{g}$. Now for all $X \in \mathfrak{g}$ we have

$$B(X,X) = Tr(\mathrm{ad}\, X\, \mathrm{ad}\, X) = \sum_{i,j} a_{ij}a_{ji} = -\sum_{i,j} a_{ij}^2 \leq 0$$

Equality in the above equation holds only if ad $X = 0$ which is true if and only if $X \in \mathfrak{z}(\mathfrak{g})$. Hence, $\mathfrak{g}'$ is semisimple and compact. Furthermore, $\mathfrak{g}' = [\mathfrak{g}', \mathfrak{g}']$ and thus $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. This completes the proof. □

Proof of Theorem 1.1.2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$. For each root α, there exists a vector $H_\alpha \in \mathfrak{h}$ such that $B(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$. Further, we can pick vectors $X_\alpha \in \mathfrak{g}_\alpha$ such that $[H, X_\alpha] = \alpha(H)X_\alpha$ and $[X_\alpha, X_{-\alpha}] = H_\alpha$. Hence, $B(X_\alpha, X_{-\alpha}) = 1$. Consequently,

$$\begin{aligned} B(X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) &= -2 \\ B(i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})) &= -2 \\ B(X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})) &= 0 \\ B(iH_\alpha, iH_\alpha) &< 0 \end{aligned}$$

Since $B(X_\alpha, X_\beta) = 0$ when $\alpha + \beta \neq 0$, it follows that B is strictly negative definite on

$$\mathfrak{g}_k = \bigoplus_\alpha \mathbb{R}iH_\alpha \oplus \bigoplus_\alpha \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \bigoplus_\alpha \mathbb{R}i(X_\alpha + X_{-\alpha})$$

Moreover, $\mathfrak{g} = \mathfrak{g}_k \oplus i\mathfrak{g}_k$. It is then readily checked that $\mathfrak{g}_k$ is a Lie algebra and hence, by the lemma, is compact. □

Using this theorem, we can now begin the constructions of Cartan involutions. These play an important role in the decompositions of the later sections and will give rise to some surprising results on the structure of non-compact groups. We start with a theorem which compares the automorphisms of $\mathfrak{g}_\mathbb{C}$ given by the real form and a compact form.

Theorem 1.1.4. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, $\mathfrak{g}$ its complexification, and $\mathfrak{u}$ a compact form of $\mathfrak{g}$. Denote by σ and τ the natural automorphisms of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}$. Then there exists an automorphism φ of $\mathfrak{g}$ such that $\varphi(u)$ is invariant under σ.*

Proof. Consider the bilinear form $B_\tau(X, Y) = -B(X, \tau Y)$ on $\mathfrak{g}$. This is strictly positive definite on $\mathfrak{u}$ as $\mathfrak{u}$ is compact. Now, let $N = \sigma\tau$. As $\tau^2 = \sigma^2 = 1$, we have that

$$B_\tau(NX, Y) = B_\tau(X, NY)$$

. and thus N is self adjoint with respect to this Hermitian form. Let $X_1, ...X_n$ be a basis of $\mathfrak{g}$ such that N is diagonal with respect to this basis. Put $P = N^2$, then P is diagonal with positive entries $\lambda_1, ..., \lambda_n > 0$. Now consider the one parameter group $P^t, t \in \mathbb{R}$ which is defined to be the diagonal matrix with entries λ_i^t. Notice that each P^t is an automorphism of $\mathfrak{g}$. If we denote by $\tau' = P^t \tau P^{-t}$, then

$$\sigma\tau' = \sigma P^t \tau P^{-t} = \sigma\tau P^{-2t} = NP^{-2t}$$
$$\tau'\sigma = P^t \tau P^{-t}\sigma = P^{2t}\tau\sigma = N^{-1}P^{2t}$$

An easy check now shows that if $t = \frac{1}{4}$ then $\sigma\tau_1 = \tau_1\sigma$. Hence, $\varphi := P^{1/4}$ has the desired properties. $\square$

Definition 1.1.5. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and $\mathfrak{g}$ is complexification. Denote by σ conjugation with respect to $\mathfrak{g}_0$. We say that a decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a **Cartan Decomposition** if there exists a compact real form $\mathfrak{g}_k$ of $\mathfrak{g}$ such that

$$\sigma(\mathfrak{g}_k) \subseteq \mathfrak{g}_k \qquad \mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_k \qquad \mathfrak{p}_0 = \mathfrak{g}_o \cap i\mathfrak{g}_k$$

By the previous theorem, we have shown that Cartan decompositions always exist for semisimple Lie algebras. It remains to check that if we have two Cartan decompositions of the same Lie algebra, then they are conjugate by an element of $\mathrm{Int}(\mathfrak{g}_0)$.

Proposition 1.1.6. *Let $\mathfrak{g}_o = \mathfrak{k}_1 \oplus \mathfrak{p}_1 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ be two Cartan decompositions of $\mathfrak{g}_0$. Then there exists an inner automorphism ψ of $\mathfrak{g}_0$ such that $\psi\mathfrak{k}_1 = \mathfrak{k}_2$ and $\psi\mathfrak{p}_1 = \mathfrak{p}_2$.*

Proof. Let $\mathfrak{u}_1$ and $\mathfrak{u}_2$ be the compact forms of $\mathfrak{g} = (\mathfrak{g}_0)_\mathbb{C}$ which give rise to the Cartan decompositions of the problem statement. By the proof of the. previous theorem, we know that $P^{1/4}$ is an automorphism which takes $\mathfrak{u}_1$ to another compact form. Notice that the killing form of $\mathfrak{g}$ is strictly negative definite on $P^{1/4}\mathfrak{u}_1$ and strictly positive definite on $i\mathfrak{u}_2$. Hence, $P^{1/4}\mathfrak{u}_1 \cap i\mathfrak{u}_2 = \{0\}$ and

$$\mathfrak{u}_2 = P^{1/4}\mathfrak{u}_1$$

. Now let σ be conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Then by construction $\sigma\tau_i = \tau_i\sigma$ and $\sigma\mathfrak{u}_i \subseteq \mathfrak{u}_i$. Since σ commutes with P^t for all t, we know P^t leaves $\mathfrak{g}_0$ invariant. As a result, P^t corresponds to a one-parameter subgroup of $\mathrm{Int}(\mathfrak{g}_0)$ given by $\exp tX$ for some $X \in \mathfrak{g}_0$. Taking $\psi = \exp \frac{1}{4}X$, we have the desired automorphism. $\square$

Remark 1.1.7. Consequently, for every pair of compact Lie subalgebras of a semisimple Lie algebra over $\mathbb{C}$, there exists a one parameter subgroup of $\mathrm{Int}(\mathfrak{g}_0)$ such that ψ^1 carries one of the compact Lie algebras to the other.

Similar to Cartan's criterion for semisimplicity, we want another characterization of the Cartan decomposition in terms of the Bilinear form associated to an automorphism.The motivation for this comes form the fact that the proofs of the above statements only ever made use of the automorphisms associated to the decompositions and not the actual decompositions themselves.

Theorem 1.1.8. *Let $\mathfrak{g}_0$ be a real Lie algebra and suppose $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$. Then the the following are equivalent:*

(a) The decomposition is a Cartan decomposition.

(b) *The mapping $\theta : k + p \mapsto k - p$, $(k \in \mathfrak{k}, p \in \mathfrak{p})$ is an automorphism of $\mathfrak{g}_0$ and the Hermitian form*

$$B_\theta(X, Y) = -B(X, \theta Y)$$

is strictly positive definite ($B < 0$ on $\mathfrak{k}$ and $B > 0$ on $\mathfrak{p}$).

This implies $\mathfrak{k}$ is a maximally compact subalgebra.

Proof. Clearly $(a) \implies (b)$ by the definition of a Cartan decomposition. For $(b) \implies (a)$, notice that as $B(\mathfrak{k}_0, \mathfrak{p}_0) = 0$ and θ is an involutive automorphism, we have that $\mathfrak{g}_k = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $(\mathfrak{g}_0)_{\mathbb{C}}$ satisfying the relations of a Cartan decomposition. It remains to show that $\mathfrak{k}_0$ is a maximally compact subalgebra.

Let $\mathfrak{g}^{\mathbb{R}}$ be $\mathfrak{g} := (\mathfrak{g}_0)_{\mathbb{C}}$ treated as a real Lie algebra. Then $\mathfrak{g}_0$ and $\mathfrak{g}_k$ are subalgebras of $\mathfrak{g}^{\mathbb{R}}$ and thus $\mathrm{Int}(\mathfrak{g}_\circ)$ and $\mathrm{Int}(\mathfrak{g}_k)$ are closed subgroups of $\mathrm{Int}(\mathfrak{g}^{\mathbb{R}})$. Furthermore, $\mathrm{Int}(\mathfrak{g}_\circ) \cap \mathrm{Int}(\mathfrak{g}_k)$ is a closed subgroup and in fact compact. The Lie algebra of this group is $\mathfrak{g}_0 \cap \mathfrak{g}_k = \mathfrak{k}_0$. Hence, $\mathfrak{k}_0$ is compactly embedded in $\mathfrak{g}_0$. Assume for the sake of contradiction that $\mathfrak{k}_0$ is not maximally compact but instead properly contained in $\mathfrak{k}_1$ another compactly embedded subalgebra of $\mathfrak{g}_0$. Then there exists a non-trivial element $X \in \mathfrak{k}_1 \cap \mathfrak{p}_0$. If we put η to be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_k$, then $\eta\mathfrak{g}_0 \subseteq \mathfrak{g}_0$ and the bilinear form B_η is symmetric and strictly positive definite on $\mathfrak{g}_0$. Since,

$$B([X, Y], \eta Z) = -B(X, [Y, \eta Z]) = B(Y, [\eta X, \eta Z])$$

we see that $\mathrm{ad}\, X$ has all its eigenvalues real and not all zero. However, this implies that $e^{n\, \mathrm{ad}\, X}$ cannot lie in a compact Lie group. This is a contradiction. □

Definition 1.1.9. An involutive automorphism θ of $\mathfrak{g}_0$ is called a **Cartan Involution** if the Hermitian form B_θ is strictly positive definite. We then identify $\mathfrak{k}_0$ with the $+1$ eigenspace and $\mathfrak{p}_0$ with the -1 eigenspace.

Remark 1.1.10. The previous theorem shows that giving a Cartan decomposition to a Lie algebra is equivalent to equipping the Lie algebra with a Cartan involution. It should be nearly obvious that if we pick a compact real form, then conjugation with respect to this form induces a Cartan decomposition for $\mathfrak{g}_0$. Hence, a Cartan involution always exists for semisimple Lie algebras. The existence of such involutions plays a key role in the following sections where we are able to decompose Lie groups using them.

1.2 Decompositions of Non-compact Groups

We begin this section by specifying the class of groups for which we shall concern ourselves for the remainder of the text.

Definition 1.2.1. A real Lie group G is said to be in the **Harish-Chandra class** if the following conditions are satisfied:

(a) $\mathfrak{g}_0$ is a reductive Lie algebra,

(b) G has finitely many connected components,

(c) The analytic subgroup G_{ss} corresponding to the Lie subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ has a finite center.

(d) $\mathrm{Ad}(G) \subseteq \mathrm{Int}(\mathfrak{g})$.

Theorem 1.2.2. *The following collections of groups are all in the Harish-Chandra class:*

(a) Compact Lie groups

(b) Semisimple Lie groups with finitely many connected components

(c) Reductive Lie groups ($\mathrm{Lie}(G) = \mathfrak{g}$ *is reductive)*

For a proof of this see [Kna05, VII.2].

Remark 1.2.3. It should be noted that the results that will be presented in the next few sections are true for all G in the Harish-Chandra class. We shall only prove them in the case that G is a non-compact connected semisimple Lie group with finite center for simplicity and will sketch how to extend the proofs for the general class.

1.2.1 The Cartan Decomposition and Maximal Compact Subgroups

In the previous section, we proved that for every semisimple Lie algebra, there exists a Cartan Involution θ and an associated decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the $+1$ and -1 eigenspaces of the involution. The goal of this subsection is to prove that we can construct a group level analog to this. The following proposition makes this precise.

Proposition 1.2.4. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and θ any Cartan involution so that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the associated Cartan decomposition. If G is any Lie group with Lie algebra $\mathfrak{g}_0$, then there exists an involutive automorphism Θ of G such that $T_e\Theta = \theta$. Moreover, the fixed point set* $\mathrm{Fix}(\Theta)$ *is a compact Lie subgroup of G with Lie algebra $\mathfrak{k}_0$ and this group is maximally compact.*

Proof. Let $\widetilde{G}$ be the universal cover of G and π the covering homomorphism. Then, any Lie algebra homomorphism of $\mathfrak{g}_0$ descends to a Lie group homomorphism. Let $\widetilde{\Theta}$ be the involution of $\widehat{G}$ corresponding to θ. By construction $T_e\widetilde{\Theta} = \theta$. Let $\widetilde{K}$ be the fixed points of $\widetilde{\Theta}$. This has Lie algebra $\mathfrak{k}_0$. Now, the kernel of π is a discrete normal subgroup and therefore contained in the center $\widetilde{Z}$. As we assume G has finite center, this implies that $\ker \pi$ is finite. Realizing G as $\widetilde{G}/\ker \pi$, we see that $\widetilde{\Theta}$ descends to an automorphism Θ of G with $T_e\Theta = \theta$. Then $\mathrm{Fix}(\Theta)$ is a closed Lie subgroup and its Lie algebra is $\{X \in \mathfrak{g}_0 : \theta(X) = X\} = \mathfrak{k}_0$. Hence, $\mathrm{Fix}(\Theta)$ is compact. Maximality now follows from Theorem 1.1.8 ($\mathfrak{k}_0$ is a maximally compact embedded subalgebra of $\mathfrak{g}_0$). $\square$

For the remainder of the text, we denote by $K := \mathrm{Fix}(\Theta)$ and refer to Θ as the **global Cartan Involution** attached to the pair $(\mathfrak{g}_0, \theta)$.

Theorem 1.2.5. *(The Cartan decomposition) Let $\mathfrak{g}_0$ be a real semisimple Lie algebra with Cartan involution θ. Denote by G and Θ the corresponding Lie group and global Cartan involution. Then the multiplication map $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism.*

Remark 1.2.6. This theorem is important because it tells us that the topology of G is completely controlled by K. That is of course because $\exp \mathfrak{p}_0$ is a Euclidean space and therefore contractible. Hence, all of the interesting properties of G should somehow arise as properties of K. This will play a massive role in understanding the motivation for $(\mathfrak{g}, K)$-modules in the next chapter. The main punchline being that $\mathfrak{g}$ becomes too local of an object to understand infinite dimensional representations as it misses all of the interesting topology of the group. By also considering K, we capture all of the necessary topological information.

Lemma 1.2.7. *Let $\mathfrak{g}_0$ be a real Lie algebra and ρ an automorphism that is diagonalizable with all positive eigenvalues $d_1, ..., d_n$. Then $\{\rho^t\}$ for $t \in \mathbb{R}$ is a one-parameter subgroup of automorphisms of $\mathfrak{g}_0$.*

Proof. Let $\mathfrak{g}_{0,d_i}$ be the d_i-eigenspace of ρ. If $X \in \mathfrak{g}_{0,d_i}$ and $Y \in \mathfrak{g}_{0,d_j}$ then

$$\rho[X, Y] = [\rho X, \rho Y] = d_i d_j [X, Y]$$

Hence, $[X, Y] \in \mathfrak{g}_{0,d_i d_j}$. As all of the eigenvalues are positive, we see that

$$\rho^t [X, Y] = (d_i d_j)^t [X, Y] = [d_i^t X, d_j^t Y] = [\rho^t X, \rho^t Y]$$

Hence, ρ^t is an automorphism for all $t \in \mathbb{R}$ and $\{\rho^t\}$ is a one-parameter subgroup. $\square$

Lemma 1.2.8. *Every one-parameter subgroup of G is given by $\exp tX$ for some unique (up to scaling) $X \in \mathfrak{g}_0$.*

See [Hel78] for details.

Proof of Theorem 1.2.5 . It suffices to prove this for $\mathrm{Ad}(G)$ with the associated involution to be $\overline{\Theta}(x) = (x^*)^{-1}$, $\overline{K} := \mathrm{Fix}(\overline{\Theta})$ and $\overline{\mathfrak{p}_0} := \mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{p}_0)$. Then, using the commuting diagram

$$\begin{tikzcd} K \times \mathfrak{p}_0 \arrow[r, "\varphi_G"] \arrow[d, "\mathrm{Ad} \times \mathrm{ad}_{\mathfrak{g}_0}"'] & G \arrow[d, "\mathrm{Ad}"] \\ \overline{K} \times \overline{\mathfrak{p}_0} \arrow[r, "\varphi_{\mathrm{Ad}(G)}"'] & \mathrm{Ad}(G) \end{tikzcd}$$

we have that the vertical maps are covering homomorphisms and thus φ_G is a diffeomorphism if and only if $\varphi_{\mathrm{Ad}(G)}$ is a diffeomorphism.

Let $x \in \mathrm{Ad}(G) \subseteq \mathrm{Int}(\mathfrak{g}_0)$ and x^* the adjoint automorphism. Then $x^* x$ is a self-adjoint for the Killing form and thus is positive definite. By lemmata 1.2.7-1.2.8 we have that $(x^* x)^t = \exp t\overline{X}$, $\overline{X} \in \mathrm{ad}\, \mathfrak{g}_0$. Therefore, put $\overline{p} = \exp \frac{1}{2} \overline{X}$ and $\overline{k} = x \overline{p}^{-1}$. Then $x = \overline{k}\overline{p}$ and $\overline{k}^* \overline{k} = 1$ by an easy computation. Applying $\overline{\Theta}$ to $\overline{k}$ we see that $\overline{\Theta}(\overline{k}) = (\overline{k}^*)^{-1} = \overline{k}$ and $\overline{\Theta}(\overline{p}) = \overline{p}^{-1}$. Hence, $\varphi_{\mathrm{Ad}\, G}$ is onto. Injectivity is obvious as each $x \in \mathrm{Ad}(G)$ determines $\overline{X}$ and hence $\overline{k}$ uniquely (Lemma 1.2.8). The map is smooth by definition. It remains to prove that the inverse mapping is smooth. To show this, consider the identification of $\mathrm{Ad}(G) \subseteq \mathrm{Int}(\mathfrak{g}_0) \hookrightarrow GL_{\dim \mathfrak{g}_0}(\mathbb{R})$. The inverse map is thus the Polar Decomposition ([Hal15, Theorem 2.17]) of the matrix image of $\mathrm{Ad}(G)$. This is smooth as each entry of the matrices in the polar decomposition is a rational function in the entires of the original matrix. Hence, the inverse is smooth and $\varphi_{\mathrm{Ad}(G)}$ is a diffeomorphism. This completes the proof. $\square$

Corollary 1.2.9. $G \simeq K$.

Proof. As $\mathfrak{p}_0$ is a real vector space it is contractible. Hence, by the Cartan decomposition for G, we see that $G \simeq K \times \mathfrak{p}_0 \simeq K \times \{*\} \cong K$. $\square$

To generalize the above results to the entire Harish-Chandra class ($\mathfrak{g}_0$ is a reductive Lie algebra), we proceed in an identical manner. Now, K is significantly more complicated but still can be realized as the fixed points of a global Cartan Involution associated to the Lie algebra. In this case, we have additional results on the structure of G. For instance, we now see that K must meet all of the components of G and thus $G = KG_0$ where G_0 is the identity component. We can still conclude that K is maximally compact by a similar argument to the one given above. What should be noted is that the above corollary still holds in this general context.

Example 1.2.10. Let $G = GL(n, \mathbb{R})$. By taking $\theta(X) = -X^T$, we have that $\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{o}(n) \oplus \mathfrak{p}$ where $\mathfrak{p}$ consists of all symmetric matrices. Then the corresponding decomposition of G is $G = O(n) \exp \mathfrak{p}$. Note that $O(n)$ has two connected components and in fact G decomposes as a semidirect product

$$G = O(n) \ltimes GL(n, \mathbb{R})^+$$

Notice that the first. decomposition is nothing more that the polar decomposition of non-singular matrices. Using this example, we can generalize this to any matrix group H by taking $H \cap O(n)$ and $H \cap \exp \mathfrak{p}$.

The final discussion of this section will be about maximal compact subgroups of G. Assuming once again that G is connected, we have the following theorem:

Theorem 1.2.11. *Let K_1 and K_2 be different maximal compact subgroups of G. Then there exists an element $g \in G$ such that $gK_1g^{-1} = K_2$. Hence, the set of all maximal compact subgroups of G form a single orbit under the natural action of G on itself.*

Proof. Let $\mathfrak{k}_1 = \mathfrak{z}_1 \oplus \mathfrak{s}_1$ be the decomposition as in Lemma 1.1.3 and $K_\mathfrak{z}$ and $K_\mathfrak{s}$ the analytic subgroups of K_1 corresponding to the respective Lie algebra. The group $K_\mathfrak{z}$ can be written as a direct product $T \times V$ where T is a torus and V is analytically isomorphic to a Euclidean space. Put $K' = K_\mathfrak{s}T$. Then K' is compact as $K_\mathfrak{s}$ is compact. It then follows immediately that $K' \cap V = \{e\}$ and thus $K_1 = K' \times V$ and K' is maximally compact in K_1.

It remains to show that the element g in the problem statement exists. This follows form [Hel78, Theorem 13.5]. Hence, every compact subgroup of G is conjugate to a subgroup of K' and $gK_1g^{-1} = K_2$ by maximality. □

To generalize this to the Harish-Chandra class, notice that combining the proof above with the general statement of the Cartan Decomposition is sufficient to show the existence of the necessary element and thus all maximal compact subgroups are conjugate.

1.2.2 Iwasawa Decomposition

This section will go through the second decomposition of a semisimple Lie group. Unlike the Cartan decomposition, this decomposition has profound consequences for integration on G. Integration enters the theory of representations when we consider intertwining operators as well as the spaces $L^2(G)$ and $L^2(\Gamma\backslash G)$. The Iwasawa decomposition will allow us to break up integrals over G in terms of integrals over compact spaces (which always converge) and integrals on abelian and nilpotent groups. Lastly, this will provide a quick proof to the Langlands decomposition of parabolic subgroups. This in turn will eventually lead to the full Langlands classification.

We start with a definition.

Definition 1.2.12. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}_0$. Then the **restricted roots** $\Sigma(\mathfrak{g}_0, \mathfrak{a})$ are the set of roots $\lambda \in \mathfrak{a}^*$ such that the space

$$\mathfrak{g}_{0,\lambda} = \{X \in \mathfrak{g} : [X, H] = \lambda(H)X, \forall H \in \mathfrak{a}, \lambda \in \mathfrak{a}^*\}$$

is non-zero.

Proposition 1.2.13. *The restricted root spaces have the following properties:*

(a) $\mathfrak{g}_0 = \mathfrak{g}_{0,0} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_{0,\lambda}$,

(b) $[\mathfrak{g}_{0,\lambda}, \mathfrak{g}_{0,\mu}] \subseteq \mathfrak{g}_{0,\lambda+\mu}$,

(c) $\theta \mathfrak{g}_{0,\lambda} = \mathfrak{g}_{0,-\lambda}$,

(d) $\mathfrak{g}_{0,0} = \mathfrak{a} \oplus \mathfrak{m}$ *where* $\mathfrak{m} = Z_{\mathfrak{k}_0}(\mathfrak{a})$.

Proof. (a) follows from the fact that $\mathfrak{a}$ is abelian and thus $\{\operatorname{ad} H : H \in \mathfrak{a}\}$ is a commuting family of self-adjoint transformations of $\mathfrak{g}_0$ and $\mathfrak{g}_0$ is an orthogonal direct sum of the eigenspaces. (b) then follows from the Jacobi identity. For (c) let $X \in \mathfrak{g}_{0,\lambda}$. Then

$$[H, \theta X] = \theta[\theta H, X] = \theta[H, X] = -\lambda(H)\theta X$$

This proves (c) and (d) follows immediately. □

Theorem 1.2.14 (Iwasawa Decomposition for Lie Algebras)**.** *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. Then there exist Lie subalgebras $\mathfrak{k}_0$ compact,$\mathfrak{a}_0$ abelian, and $\mathfrak{n}_0$ nilpotent such that*

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$$

Proof. Let $(\mathfrak{g}, \theta)$ by a semisimple Lie algebra together with a Cartan involution. Put $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition and $\mathfrak{h}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$. Pick an ordering on $\mathfrak{h}_{\mathfrak{p}}^*$ and let $\mathfrak{n} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$. Since $\mathfrak{h}_{\mathfrak{p}}$ is θ-invariant and maximal abelian, we have that

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) + \mathfrak{h}_{\mathfrak{p}}$$

Now if $X \in \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$ we can write it as $X = X + \theta(X) - \theta(X)$. This decomposition has $X \in \mathfrak{k} \oplus \mathfrak{n}$. Therefore, we have a decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{h}_{\mathfrak{p}} + \mathfrak{n}$$

Applying θ we conclude that this decomposition is direct. □

There is an associated group decomposition which requires a few technical lemmata to prove. We present them with no proof as they are nearly obvious.

Lemma 1.2.15. *Let H be an analytic group with Lie algebra $\mathfrak{h}$. Suppose that $\mathfrak{h}$ decomposes as a vector space direct sum $\mathfrak{s} \oplus \mathfrak{t}$. If S and T are analytic subgroups with the respective Lie algebras, then the multiplication map $S \times T \to H$ is everywhere regular.*

Lemma 1.2.16. *There is a basis of $\mathfrak{g}_0$ such that the matrices of* $\operatorname{ad} \mathfrak{g}_0$ *have the following properties:*

(a) $\operatorname{ad} \mathfrak{k}$ *consists of skew-symmetric matrices.*

(b) $\operatorname{ad} \mathfrak{a}$ *consists of diagonal matrices.*

(c) $\operatorname{ad} \mathfrak{n}$ *consists of strictly upper-triangular matrices.*

Theorem 1.2.17 (The Iwasawa Decomposition)**.** *Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}_0$. If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is an Iawasawa decomposition, and A and N are analytic subgroups of G corresponding to the respective Lie algebras, then the multiplication map $K \times A \times N \to G$ is a diffeomorphism. Further, A and N are simply connected.*

Proof. Similar to the proof of the group level Cartan decomposition, we shall prove the decomposition for $\operatorname{Ad}(G) \subseteq \operatorname{Int}(\mathfrak{g})$ and then lift the result to G. Let $\overline{K} = \operatorname{Ad}_{\mathfrak{g}_o}(K)$, $\overline{A} = \operatorname{Ad}_{\mathfrak{g}_o}(A)$, and $\overline{N} = \operatorname{Ad}_{\mathfrak{g}_o}(N)$. In the basis of the previous lemma, $\overline{K}$ consists of rotation matrices, $\overline{A}$ consists of diagonal matrices with. positive entries, and $\overline{N}$ consists of strictly upper triangular matrices. As the diagonal subgroup of $\operatorname{Aut}(()\mathfrak{g}_0)$ is simply connected and abelian, we see that $\overline{A}$ is simply connected and abelian. Similarly, $\overline{N}$ is simply connected. Further, as they are analytic subgroups, they are closed in $\operatorname{Ad}(G)$.

Consider the canonical map $\overline{A} \times \overline{N} \to \operatorname{Ad}(G)$. This map is injective as we can recover $\bar{a}$ from the diagonal entries of the matrix image. Therefore, it is isomorphic to a subgroup $\overline{AN}$. This subgroup is closed as both components are closed. Clearly, it has Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ and by Lemma 1.2.15, the map is a diffeomorphism.

Lastly, we see that $\overline{K} \cap \overline{AN} = \{1\}$ as the only rotation matrices with positive eigenvalues is 1. Therefore, the map $\overline{K} \times \overline{AN} \to \operatorname{Ad}(G)$ is injective and applying Lemma 1.2.15 again, we see that it is a diffeomorphism. This completes the proof for $\operatorname{Ad}(G)$. Now, using the commutative diagram

$$\begin{tikzcd}
K \times A \times N \arrow[r, "\iota_G"] \arrow[d, "\operatorname{Ad} \times \operatorname{Ad} \times \operatorname{Ad}"'] & G \arrow[d, "\operatorname{Ad}"] \\
\overline{K} \times \overline{A} \times \overline{N} \arrow[r, "\iota_{\operatorname{Ad}(G)}"'] & \operatorname{Ad}(G)
\end{tikzcd}$$

we see that the result holds for G as well. ☐

Example 1.2.18. (a) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. $SO(n) \hookrightarrow SL(n, \mathbb{R})$ is a maximal compact subgroup and therefore $\mathfrak{so}(n)$ is the corresponding compact lie algebra. Let $\mathfrak{a}$ be the traceless diagonal matrixes and $\mathfrak{n}$ be strictly upper triangular matrices. Then

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{a} \oplus \mathfrak{n}$$

We can equivalently realize this on the group level as $SL(n, \mathbb{R}) = SO(n) \cdot T \cdot N$ where N is upper triangular matrices and T is the maximal torus. Notice that this is equivalent to the Gram-Schmidt orthogonalization of a matrix in $\mathfrak{sl}(n)$.

Now lets consider the Cartan decomposition of $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{p}_0$ where $\mathfrak{p}_0$ are symmetric matrices. Notice that $\mathfrak{so}(n)$ appears in both decompositions yet for the Cartan decomposition we have no lie algebra structure on $\mathfrak{p}_0$. This should not be surprising however as both decompositions are equivalences as vector spaces.

(b) Now consider $\mathfrak{sp}(2n, \mathbb{C})$. We have that

$$\mathfrak{k} = \left\{ \begin{pmatrix} U & V \\ -\bar{V} & \bar{U} \end{pmatrix} : U \text{ skew-Hermitian}, V \text{ symmetric} \right\}$$

Similar to $\mathfrak{sl}_n$ we have $\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A \text{ real diagonal matrix} \right\}$ which are the diagonal matrices and the nilpotent lie algebra are all upper triangular matrices, but now we can decompose them further into

$$\mathfrak{n} = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ 0 & -Z_1^T \end{pmatrix} : Z_1 \text{ strictly upper triangular}, Z_2 \text{ symmetric} \right\}$$

Then $\mathfrak{sp}(2n, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

We shall end this chapter with some beautiful corollaries of the Iwasawa decomposition.

Definition 1.2.19. We call $\mathfrak{c} \subseteq \mathfrak{g}_0$ a **Cartan Subalgebra** if $\mathfrak{c}_\mathbb{C}$ is a Cartan subalgebra of $(\mathfrak{g}_0)_\mathbb{C}$.

Corollary 1.2.20. *If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} = Z_\mathfrak{k}(\mathfrak{a})$, then $\mathfrak{h} = \mathfrak{a}_0 \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_0$.*

Proof. It suffices to show that $\mathfrak{h}_\mathbb{C}$ is maximal abelian in $(\mathfrak{g}_0)_\mathbb{C}$ and that $\mathrm{ad}_{(\mathfrak{g}_0)_\mathbb{C}}(\mathfrak{h}_\mathbb{C})$ is simultaneously diagonalizable. Clearly, the first condition is satisfied as any element which commutes with $\mathfrak{h}_\mathbb{C}$ necessarily commutes component-wise. In the basis of Lemma 1.2.16, each $\mathfrak{t}$ is skew-symmetric and hence diagonalizable. As every element of $\mathfrak{a}$ is diagonal, $\mathrm{ad}_{(\mathfrak{g}_0)_\mathbb{C}}(\mathfrak{h}_\mathbb{C})$ is simultaneously diagonalizable. This completes the proof. ☐

Corollary 1.2.21. *All of the roots of $\mathfrak{h} = \mathfrak{a}_0 \oplus \mathfrak{t}$ are real valued on $\mathfrak{a}_0 \oplus i\mathfrak{t}$.*

Corollary 1.2.22. *If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are distinct maximal abelian subspaces of $\mathfrak{p}_0$, then there exists an element $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a}_1 = \mathfrak{a}_2$. Consequently, $\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}$.*

Proof. There are finitely many restricted roots with respect to $\mathfrak{a}_1$ and thus the union of the kernels cannot exhaust $\mathfrak{a}_1$. We can fine $H_1 \in \mathfrak{a}$ such that $Z_{\mathfrak{p}_0}(H) = \mathfrak{a}_1$ and similarly for $\mathfrak{a}_2$. By the compactness of $\mathrm{Ad}(K)$, we can find an element $k \in K$ such that $B(\mathrm{Ad}(k)H_2, H_1)$ is minimized. For any $Q \in \mathfrak{k}$, we have that

$$r \mapsto B(\mathrm{Ad}(\exp rZ)\,\mathrm{Ad}(k)H_2, H_1)$$

is a smooth function of r and is minimized at $r = 0$. By differentiating, we see that $0 = B(Z, [\mathrm{Ad}(k)H_2, H_1])$. Since, $B(\mathfrak{k}, \mathfrak{p}) = 0$ and B is non-degenerate, we see that $[\mathrm{Ad}(k)H_2, H_1] = 0$. Thus $\mathrm{Ad}(k)H_2 \in Z_{\mathfrak{p}_0}(H_1) = \mathfrak{a}_1$. Since, $\mathfrak{a}_1$ is abelian, it implies that $\mathfrak{a}_1 \subseteq \mathrm{Ad}(k)\mathfrak{a}_2$ and by maximality they are equal.

Now let $X \in \mathfrak{p}_0$. Then we can extend $\mathbb{R}X$ to a maximal abelian subspace. By what we have just shown, there is an element k such that $X \in \mathrm{Ad}(k)\mathfrak{a}$. Hence, $\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}$. ☐

Theorem 1.2.23 (The KAK decomposition)**.** *Let G be a connected semisimple Lie group, and $K \times \mathfrak{p}_0$ a Cartan Decomposition. Then there for any maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}_0$ and associated analytic subgroup A, we have the decomposition $G = KAK$.*

Proof. By the Cartan decomposition, we see that $G/K = \exp \mathfrak{p}_0 = \exp \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}$ by Corollary 1.2.22. Therefore, for any $P \in \mathfrak{p}_0$, we have that $\exp P = gk_1 = \exp \mathrm{Ad}(k)a = \mathrm{Ad}(k) \exp a = k \exp ak^{-1}$. Therefore, $g = k \exp a(k_1 k)^{-1}$. As $A = \exp \mathfrak{a}$, we see that $G = KAK$. ☐

As wel will see in the second half of Chapter 2, the KAK decomposition becomes incredibly important for understanding spherical vectors and spherical principal series representations. These are defined in terms of K bi-invariant functions $f \in C_c^\infty(K\backslash G/K)$. For this reason, we can turn integrals of these functions over G into integrals over A by the above decomposition. This makes understanding them far easier. Also, a theorem of Gel'fand tells us. that this ring is commutative which again simplifies the situation mightily.

Before we finish this section, we want to know that if we have two different Iwasawa decompositions, then they are conjugate. That is if $\mathfrak{a}$ and $\mathfrak{a}'$ are different maximal ablian subspaces of $\mathfrak{p}_0$, then we would like to find an element of $\mathrm{Int}(\mathfrak{g}_0)$ which carries $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ to $\mathfrak{k} \oplus \mathfrak{a}' \oplus \mathfrak{n}'$.

Proposition 1.2.24. *Every Iwasawa decomposition is conjugate by an inner automorphism on both the Lie algebra and group level.*

Proof. It remains to show that there are elements in $\mathrm{Ad}(K)$ which conjugate the nilpotent parts of the Iwasawa decompositions. This follows from the fact that $\Sigma(\mathfrak{g}_0, \mathfrak{a})$ is an abstract root system in $\mathfrak{a}^*$. Then, each choice of simple system is conjugate by an element of the Weyl group $W(\Sigma)$. In particular, we can realize these elements as sitting inside $N_K(\mathfrak{a})$. Hence, each $\mathfrak{n}$ is conjugate via $\mathrm{Ad}(K)$. Combining this with Corollary 1.2.22, we see that every Iwasawa decomposition of $\mathfrak{g}_0$ is conjugate. □

The final application of this decomposition is to integration on G. The key to understanding a majority of the representation theory of reductive, semisimple, or compact Lie groups is the existence of a *Haar Measure*. This is a left invariant Borel measure on G. The existence of such a measure implies, as an example, that all representations of compact Lie groups can be taken to be unitary without a loss of generality. Additionally, combined with the Iwasawa decomposition, we get a variety of strong results. This will play a key role in the proof of the Borel-Weil theorem. Let us first show that such a measure exists.

Let G be a Lie group of dimension n with Lie algebra $\mathfrak{g}$. Then as $T_1(G) = \mathfrak{g}$ and there is an isomorphism $\mathfrak{g} \to \Gamma_L(G, TG)$ the set of left-invariant smooth vector fields on G. From this we conclude that G is parallelizable. For this reason, we know that there exists an $n-$form $\omega \in \Omega^n(G)$ such that ω is positive relative to a chosen atlas on G, is nowhere vanishing, and is left-invariant. Further, by the Riesz Representation theorem, there exists a Borel measure $d\mu_\omega$ on G such that $\int_G f\omega = \int_G f d\mu_\omega$ for all $f \in C_c(G)$.

Lemma 1.2.25. *$d\mu_\omega$ is left invariant in the sense that $d\mu_\omega(L_g E) = d\mu_\omega(E)$ for all Borel sets $E \subseteq G$ and all $g \in G$.*

Proof. As ω is left-invariant, we know that $L_g^*\omega = \omega$. Therefore, we have that

$$\int_G f\omega = \int_G f(gx)L_g^*\omega = \int_G f(gx)d\mu_\omega(x) = \int_G f(x)d\mu_\omega(x)$$

Hence, $d\mu_\omega$ is left-invariant. If $K \subseteq G$ is compact, we apply the above integral formula to all $f \geq 1_K$. Taking the infimum over these. functions we see that $d\mu_\omega(L_g^*K) = d\mu_\omega(K)$. Since G has a countable base, $d\mu_\omega$ is regular and the lemma follows. □

Definition 1.2.26. A left-invariant, positive, Borel measure on G is called a **left Haar measure**.

Proposition 1.2.27. *Every left Haar measure on G is proportional.*

Proof. See [Kna05, Theorem 8.23]. ☐

We could have equivalently defined **right Haar measures**. For most groups these are different from the left Haar measures. Let $d_l x$ denote a left Haar measure and $d_r x$ a right Haar measure. Notice that L_g and R_g commute with one another. Then, for any $t \in G$, the measure $d_l(\cdot t)$ is a left Haar measure. For this reason, we get a function $\Delta : G \to \mathbb{R}^+$ called the **modular homomorphism** which satisfies

$$d_l(\cdot t) = \Delta^{-1}(t) d_l(\cdot)$$

This is a smooth function.

Lemma 1.2.28. $\Delta(t) = 1$ *for all* $t \in K$ *a compact subgroup of* G.

Proof. As Δ is smooth, $\Delta(K)$ is a compact subgroup of $\mathbb{R}^+$. Therefore $\Delta(K) = \{1\}$. ☐

Definition 1.2.29. A Lie group G is called **unimodular** if $\Delta = 1$. Equivalently, if $d_r(x) = d_l(x)$.

We now want to know what groups are unimodular. Then, when integration arises on these groups we do not have to worry about the choice of Haar measure.

Theorem 1.2.30. *The following groups are unimodular:*

(a) Compact groups

(b) semisimple groups

(c) Reductive groups

Lemma 1.2.31. *If* G *is any Lie group, then the modular function is given by* $\Delta(t) = |\det \mathrm{Ad}(t)|$.

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. If $X \in \mathfrak{g}$, denote by $\widetilde{X}$ the left-invariant vector field on G defined by X. Then by definition, for any $h \in C^\infty(G)$

$$(dR_{t^{-1}})_p \widetilde{X}_p h = \widetilde{\mathrm{Ad}(t)} X h(pt^{-1}$$

and therefore as operators $dR_{t^{-1}})_p \widetilde{X}_p = \mathrm{Ad}(t) X_{pt^{-1}}$. Now, if $\omega \in \Omega_L^{top}(G)$ we have by the previous relation

$$R_{t^{-1}}^* \omega = \det \mathrm{Ad}(t) \omega$$

Now assuming that ω is the positive choice of orientation, we see that $\det \mathrm{Ad}(t)$ determines the sign of the form $R_{t^{-1}}^* \omega$. If $\det Ad(t) > 0$, then

$$\begin{aligned}
\det \mathrm{Ad}(t) \int_G f\omega &= \int_G f R_{t^{-1}}^* \omega \\
&= \int_G (f \circ R_t) \omega \\
&= \int_G f(xt) d_l(x) \\
&= \int_G f(x) d_l(xt^{-1} \\
&= \Delta(t) \int_G f(x) d_l(x)
\end{aligned}$$

for all $f \in C_c^\infty(G)$. Hence, $\Delta(t) = \det \mathrm{Ad}(t)$. If $\det Ad(t) < 0$, then the second line has an additional negative sign. Therefore $\Delta(t) = |\det \mathrm{Ad}(t)|$ for all $t \in G$. ☐

Lemma 1.2.32. *For any $a \in A$, we have that $\Delta(a) = e^{2\rho_A \log a}$.*

Proof. Notice that $\Delta(an) = |\det \mathrm{Ad}_{\mathfrak{a}\oplus\mathfrak{n}}(an)|$ by the previous lemma. On each $\mathfrak{g}_{0,\lambda}$, a acts by $e^{\lambda \log a}$. As $\mathfrak{a} \subseteq \mathfrak{g}_{0,0}$, we see that a acts by 0 on $\mathfrak{a}$. Hence, a acts on $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_{0,\lambda}$ by $e^{2\rho_A \log a}$. □

Proposition 1.2.33. *Let G be a Lie group, and let S and T closed subgroups such that $S \cap T$ is compact, multiplication is an open map, and the subgroup ST exhausts G up to a set of measure 0. Let Δ_T and Δ_G denote the modular functions of T and G respectively. Then the left Haar measure on G can be normalized such that*

$$\int_G f(x) d_l x = \int_{S\times T} f(st) \frac{\Delta_T(t)}{\Delta_G(t)} d_l s d_l t$$

We omit the proof as it is not illuminating for this discussion. From this proposition, we can conclude the following:

Theorem 1.2.34. *Let G be a semisimple Lie group and dg a Haar measure on G. Let $G = KAN$ be an Iwasawa decomposition and f any measurable function on G. Then there exists a decomposition of the Haar measure for G such that $dg = dkdadn$ and*

$$\int_G f(g) dg = \int_N \int_A \int_K f(kan) \Delta_{AN}(a) dk\ da\ dn$$

where dk, da, dn denote the Haar measures on K, A, and N respectively. If instead we write $G = ANK$, then

$$\int_G f(g) dg = \int_K \int_N \int_A f(ank) da\ dn\ dk$$

Proof. Let $K = S$ and $AN = T$ in the previous proposition. Then we have that $dg = dkd_r(an) = \Delta_{AN}(an) dk\ da\ dn = e^{2\rho_A \log a} dk\ da\ dn$ by the proposition. For the second decomposition, it follows that $d_l(an) = da\ dn$ and thus $dg = da\ dn\ dk$. This completes the proof. □

Chapter 2

Analytic Representation Theory

Most of this section will follow [VJ81], [KVJ95],and [Kna86].

2.1 Finite-dimensional Representations

We begin this chapter with some generalities about harmonic analysis on semisimple Lie groups. In particular, the goal of this section will be to get to the Borel-Weil theorem by way of parabolic induction. This idea generalizes to infinite dimensional representations in a fundamental way (as the Principal Series Representations of section 2.3). The Borel-Weil theorem is the case of compact groups and inducing representations from parabolic subgroups. In the later sections, we will only consider non-compact groups and thus the induced representations will be infinite dimensional.

Remark 2.1.1. For this entire section we fix a real semisimple Lie algebra $\mathfrak{g}_0$ and its complexification will be denoted by $\mathfrak{g}$. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_0$, denote by $\Delta := \Delta(\mathfrak{g}, \mathfrak{t}_{\mathbb{C}})$ the set of roots. By making a choice of simple system $\Pi \subseteq \Delta$, we have a notion of positive roots. Let

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \qquad\qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$$

be the nilpotent subalgebras corresponding to the positive and negative root spaces.

2.1.1 Parabolic Subalgebras: the Complex case

For the first part of this section we shall study parabolic subalgebras of $\mathfrak{g}$ which contain the standard Borel subalgebra $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$. Then our focus will move to understanding general parabolic subalgebras and parabolic subgroups for reductive Lie groups. As all of this section is true, regardless of the choice of real form, we shall denote $\mathfrak{t}_{\mathbb{C}}$ by $\mathfrak{h}$ to remain consistent with the standard notation in the literature.

Consider the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\Delta} \mathfrak{g}_\alpha$. The Borel subalgebra is in some sense the smallest non-abelian subalgebra which contains some interesting information about the Lie algebra $\mathfrak{g}$.

Definition 2.1.2. A Lie subalgebra $\mathfrak{q}$ which contains $\mathfrak{b}$ is called a **parabolic subalgebra**. As each of the root spaces is 1-dimensional, $\mathfrak{q}$ necessarily is of the form

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha \tag{2.1}$$

where Γ is a subset of Δ containing Δ^+. The extreme cases are $\Gamma = \Delta$ where $\mathfrak{q} = \mathfrak{g}$ and $\Gamma = \Delta^+$ where $\mathfrak{q} = \mathfrak{b}$. For this reason, we call $\mathfrak{b}$ the **standard minimal parabolic subalgebra**.

We want to obtain further examples of parabolic subalgebras. To do this, fix $\Pi' \subseteq \Pi$ and put

$$\Gamma_{\Pi'} = \Delta^+ \cup \{\alpha \in \operatorname{Span} \Pi'\}$$

By construction, we see that the direct sum of the root spaces associated to Γ is indeed a Lie subalgebra. What we would like to know is that if this construction exhausts all parabolic subalgebras containing $\mathfrak{b}$. The following proposition gives a positive answer to this.

Proposition 2.1.3. *The parabolic subalgebras $\mathfrak{q}$ containing $\mathfrak{b}$ are parametrized by the set of subsets of simple roots. The correspondence is given by $\Pi' \mapsto \mathfrak{q}(\Pi') = \mathfrak{h} \oplus \bigoplus_{\Gamma_{\Pi'}} \mathfrak{g}_\alpha$.*

Proof. From the discussion above, we know that all parabolic subalgebras containing $\mathfrak{b}$ are necessarily of the form 2.1. Notice that Γ in this decomposition is generated by Δ^+ and the set of simple roots contained in the span of $\Gamma \cap -\Gamma$. Put $\Pi'(\mathfrak{q})$ to be this set of simple roots. Then in one direction, the correspondence is given by $\mathfrak{q} \mapsto \Pi'(\mathfrak{q})$. Therefore, the main statement we need to prove is:

Claim 2.1.4. *The assignments $\mathfrak{q} \mapsto \Pi'(\mathfrak{q})$ and $\Pi' \mapsto \mathfrak{q}(\Pi')$ are inverse to one another.*

To show that $\Pi'(\mathfrak{q}(\Pi')) = \Pi'$, notice that

$$\Gamma_{\Pi'} \cap -\Gamma_{\Pi'} = (\Delta^+ \cap -\Delta^+) \cup \operatorname{Span} \Pi' = \operatorname{Span} \Pi'$$

The simple roots are then the elements of Π'. Hence, $\Pi'(\mathfrak{q}(\Pi')) = \Pi'$.

To show that $\mathfrak{q}(\Pi'(\mathfrak{q}_1)) = \mathfrak{q}_1$, we argue by induction on the level of a root. Any positive element of $\operatorname{Span} \Pi'(\mathfrak{q})$ is a member of $\Gamma_{\mathfrak{q}}$ (the set of roots defining $\mathfrak{q}_1$). If $-\alpha$ is a negative root in the span of $\Pi'(\mathfrak{q})$, we can write this as a non-positive integral combination of members of $\Pi'(\mathfrak{q})$. Let $\alpha = \sum n_i \alpha_i$ be such a decomposition. If $\sum n_i = 1$, then α is simple and $-\alpha \in \Gamma_{\mathfrak{q}}$ by definition. Now, if $\sum n_i > 1$, there exist $\beta, \gamma \in \Delta^+$ such that $\alpha = \beta + \gamma$. By the induction hypothesis, $-\beta, -\gamma \in \Gamma_{\mathfrak{q}_1}$ and hence, so is $-\alpha$. This proves that $\Gamma_{\Pi'(\mathfrak{q})} \subseteq \Gamma_{\mathfrak{q}_1}$ and thus $\mathfrak{q}(\Pi'(\mathfrak{q}_1)) \subseteq \mathfrak{q}_1$.

Now, applying the same exact argument for negative roots $-\alpha \in \Gamma_{\mathfrak{q}} \cap -\Gamma_{\mathfrak{q}}$, we see that $\Gamma_{\mathfrak{q}} \cap -\Gamma_{\mathfrak{q}} \subseteq \Gamma_{\Pi'(\mathfrak{q}_1)}$. Hence, $\Gamma_{\Pi'(\mathfrak{q}_1)} = \Gamma_{\mathfrak{q}_1}$ and thus $\mathfrak{q}_1 = \mathfrak{q}(\Pi'(\mathfrak{q}_1))$. □

Given the above proposition, we can decompose any parabolic $\mathfrak{q}$ containing $\mathfrak{b}$ into the following two subspaces:

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha \qquad \mathfrak{u} = \bigoplus_{\substack{\alpha \in \Gamma \\ \alpha \notin -\Gamma}} \mathfrak{g}_\alpha \tag{2.2}$$

The following proposition characterizes $\mathfrak{l}$ and $\mathfrak{u}$.

Proposition 2.1.5. *[Levi Factors] Let $\mathfrak{q}$ be a parabolic subalgebra containing $\mathfrak{b}$ and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ as above.*

(a) $\mathfrak{l}$ and $\mathfrak{u}$ are subalgebras of $\mathfrak{q}$, and $\mathfrak{u}$ is an ideal.

(b) $\mathfrak{u}$ is nilpotent.

(c) $\mathfrak{l}$ is reductive with center $\mathfrak{h}'' = \bigcap_{\alpha \in \Gamma \cap -\Gamma} \ker \alpha$. It semisimple part $\mathfrak{l}_{ss}$ has a root space decomposition

$$\mathfrak{l}_{ss} = \mathfrak{h}' \oplus \bigoplus_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha$$

where $\mathfrak{h}' = \sum_{\alpha \in \Gamma \cap -\Gamma} \mathbb{C} H_\alpha$.

Proof. Let $\mathfrak{q}$ be as in the previous proposition. Then (a) is clear. For (b) we have that $\mathfrak{u} \subseteq \mathfrak{n}$ and thus $\mathfrak{u}$ is nilpotent.

Therefore, we need to prove (c). Let $\mathfrak{h}_0$ be a real form of $\mathfrak{h}$ on which all roots are real valued. Then $\mathfrak{h}_0' = \mathfrak{h}_0 \cap \mathfrak{h}'$ and $\mathfrak{h}_0'' = \mathfrak{h}_0 \cap \mathfrak{h}''$ are real forms of $\mathfrak{h}'$ and $\mathfrak{h}''$ respectively. Then its clear that $\mathfrak{h}_0'$ and $\mathfrak{h}_0''$ are orthogonal complements under the Killing form. Hence, $\mathfrak{h}_0 = \mathfrak{h}_0' \oplus \mathfrak{h}_0''$ and $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. If $\mathfrak{l}_{ss}$ is as in the statement of the proposition, then $\mathfrak{l} = \mathfrak{h}'' \oplus \mathfrak{l}_{ss}$. It is clear that $\mathfrak{h}''$ and $\mathfrak{l}_{ss}$ are ideals and $\mathfrak{h}''$ is central. Thus it suffices to show that $\mathfrak{l}_{ss}$ is semisimple.

Let B' be the Killing form on $\mathfrak{l}_{ss}$. It suffices to show that B' is non-degenerate. For each $\alpha \in \Gamma \cap -\Gamma$, pick root vectors so that $[E_\alpha, E_{-\alpha}] = H_\alpha$ and $B(E_\alpha, E_{-\alpha}) = 1$. We will show that $B'(E_\alpha, E_{-\alpha}) > 0$ and B' is positive definite on $\mathfrak{h}_0' \times \mathfrak{h}_0'$. It follows from the root space. decomposition that $\operatorname{ad} E_\alpha \operatorname{ad} E_{-\alpha}$ acts with a non-negative eigenvalue on any $\mathfrak{g}_\beta$. Therefore, the trace of $\operatorname{ad} E_\alpha \operatorname{ad} E_{-\alpha}$ is positive and thus $B'(E_\alpha, E_{-\alpha}) > 0$.

Now we turn our attention to $\mathfrak{h}_0'$. If $H \in \mathfrak{h}_0'$ then $B'(H,H) = \sum_{\alpha \in \Gamma \cap -\Gamma} \alpha(H)^2$ with each term ≥ 0. To get 0 one of the $\alpha(H) = 0$ which implies that $H \in \mathfrak{h}''$. As $\mathfrak{h}' \cap \mathfrak{h}'' = 0$, we see that $H = 0$ and thus B' is non-degenerate and $\mathfrak{l}_{ss}$ is semisimple. □

We call $\mathfrak{l}$ the **Levi factor** and $\mathfrak{u}$ the **nilpotent radical**. The decomposition above is called the **Levi decomposition** of the Lie algebra. The Levi factor depends on the choice of $\mathfrak{h}$ as well as $\mathfrak{q}$ whereas $\mathfrak{u}$ can be defined intrinsically using the Killing form. Some nice features of the Levi decomposition are as follows.

Put $\mathfrak{q}^- = \mathfrak{l} \oplus \theta\mathfrak{u} = \theta\mathfrak{q}$ This is the *opposite* parabolic associated to $\mathfrak{q}$ and contains the Borel subalgebra $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^- = \mathfrak{h} \oplus \theta\mathfrak{n}$. It then follows immediately that

$$\mathfrak{l} = \mathfrak{q} \cap \mathfrak{q}^- \qquad\qquad \mathfrak{g} = \theta\mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{u}$$

Lemma 2.1.6. *Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{q}) = \{X \in \mathfrak{g} : [X, \mathfrak{q}] \subseteq \mathfrak{q}\}$.*

Proof. Clearly $\mathfrak{q} \subseteq N_{\mathfrak{g}}(\mathfrak{q})$ and thus it suffices to prove the reverse inclusion. Let Π' and Γ' be the defining set of roots for $\mathfrak{q}$ given by Proposition 2.1.3. As $\mathfrak{b} \subseteq \mathfrak{q} \subseteq N_{\mathfrak{g}}(\mathfrak{q})$, we see that $N_{\mathfrak{g}}(\mathfrak{q})$ is a parabolic subalgebra of $\mathfrak{g}$. Therefore, there exists a set of simple roots Π'' and an associated collection Γ'' such that

$$N_{\mathfrak{g}}(\mathfrak{q}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma''} \mathfrak{g}_\alpha$$

Suppose for the sake of contradiction that Γ'' is strictly greater than Γ' and $\alpha \in \Gamma'' - \Gamma'$. Pick a root vector $X_\alpha \in \mathfrak{g}_\alpha \subseteq N_{\mathfrak{g}}(\mathfrak{q})$. Then for all $H \in \mathfrak{h} \subseteq \mathfrak{q}$, $[X_\alpha, H] \in \mathfrak{q}$ by the definition of the normalizer. However, by construction $[X_\alpha, H] = \alpha(H) X_\alpha$. This is a contradiction as X_α not in $\mathfrak{q}$ as we assume $\alpha \notin \Gamma'$. Hence, $\Gamma' = \Gamma''$ and $\mathfrak{q} = N_{\mathfrak{g}}(\mathfrak{q})$. □

Lemma 2.1.7. *Let $\mathfrak{w} \subseteq \mathfrak{g}$ be any Lie subalgebra and G a complex Lie group with Lie algebra $\mathfrak{g}$. Then the Lie algebra of the Lie subgroup $N = N_G(\mathfrak{w})$ is $N_{\mathfrak{g}}(\mathfrak{w})$. Similarly, the Lie algebra of $Z_G(\mathfrak{w})$ is $Z_{\mathfrak{g}}(\mathfrak{w})$.*

Proof. Let $W \in \mathfrak{w}$ and $X \in N_{\mathfrak{g}}(\mathfrak{w})$. Then

$$\begin{aligned} \frac{d}{dt} \operatorname{Ad}(\exp(tX))(W)\Big|_{t=0} &= \frac{d}{dt} \exp(\operatorname{ad}(tX))(W)\Big|_{t=0} \\ &= \frac{d}{dt} \exp(t \operatorname{ad}(X))(W)\Big|_{t=0} \\ &= \operatorname{ad}(X) \exp(t \operatorname{ad}(X))|_{t=0}(W) \\ &= \operatorname{ad}(X)(W) \in \mathfrak{w} \end{aligned}$$

Hence, $N_{\mathfrak{g}}(\mathfrak{w})$ is the Lie Algebra of $N_G(\mathfrak{w})$. Similarly, $Z_{\mathfrak{g}}(\mathfrak{w})$ is the Lie algebra of $Z_G(\mathfrak{w})$. □

Corollary 2.1.8. *Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. Then $N_G(\mathfrak{q})$ is a Lie subgroup with Lie algebra $\mathfrak{q}$.*

Consider again the decomposition of $\mathfrak{l}$ as in Proposition 2.1.5. As $\mathfrak{h}'' = \mathfrak{z}(\mathfrak{l})$, then $Z_{\mathfrak{g}}(\mathfrak{h}'') \supseteq \mathfrak{l}$ by definition. In fact, the centralizer is generated by $\mathfrak{h}$ and all roots which vanish on $\mathfrak{h}''$. Hence,

$$\mathfrak{l} = Z_{\mathfrak{g}}(\mathfrak{h}'') \tag{2.3}$$

Now, by the previous Lemma, we have that $\mathfrak{l}$ is the Lie algebra to $Z_G(\mathfrak{h}'')$. As in the proof of the Iwasawa decomposition, we see that if $U = \exp \mathfrak{u}$, then

$$Z_G(\mathfrak{h}'') \times U \xrightarrow{\sim} Q = N_G(\mathfrak{q})$$

We can further understand $Z_G(\mathfrak{h}'')$. Notice that as $\mathfrak{l} = \mathfrak{q} \cap \theta\mathfrak{q}$ we can nearly conclude that

$$Z_G(\mathfrak{h}'') \stackrel{?}{=} Q \cap \Theta Q \tag{2.4}$$

One obstruction to this however is that we do not know how elements of the centralizer interact with Q. To fully conclude that this is an equality, we need the following Proposition.

Proposition 2.1.9. *$\mathfrak{l} = N_{\mathfrak{q}}(\mathfrak{l})$ and $L := N_Q(\mathfrak{l})$ is a closed subgroup of Q with Lie algebra $\mathfrak{l}$. Further, $L = Q \cap \Theta Q$.*

Proof. Clearly, $\mathfrak{l} \subseteq N_{\mathfrak{q}}(\mathfrak{l})$. Suppose for the sake of contradiction that $\exists \beta \in \Gamma, \beta \notin -\Gamma$ with $[\mathfrak{g}_\beta, \mathfrak{l}] \subseteq \mathfrak{l}$. Then $\mathfrak{g}_\beta \subseteq \mathfrak{u}$. As $\mathfrak{u}$ is an ideal, we have that $[\mathfrak{g}_\beta, \mathfrak{l}] \subseteq \mathfrak{u}$. As $\mathfrak{u} \cap \mathfrak{l} = 0$, we conclude that $[\mathfrak{g}_\beta, \mathfrak{l}] = 0$. In each $\mathfrak{g}_\beta$, there exists a root vector X_β such that for all $H \in \mathfrak{h}$,

$$[X_\beta, H] = \beta(H) X_\beta$$

For this to be 0, it would imply that $\beta(H) = 0$ for all $H \in \mathfrak{h}$, and thus $\beta = 0$. This is a contradiction and hence, $\mathfrak{l} = N_{\mathfrak{q}}(\mathfrak{l})$. Applying the above Lemma, we see that $L = N_Q(\mathfrak{l})$ is a closed subgroup and has Lie algebra $\mathfrak{l}$. Now, as Q and ΘQ are transversal submanifolds, we have that the Lie algebra of their intersection is $\mathfrak{q} \cap \theta\mathfrak{q} = \mathfrak{l}$.

It remains to show that $L = Q \cap \Theta Q$. Let $x \in G$. If $x \in Q \cap \Theta Q$, then $\operatorname{Ad}(x)\mathfrak{l} \subseteq \mathfrak{q}$ and $\operatorname{Ad}(x)\mathfrak{l} \subseteq \theta\mathfrak{q}$. Hence, $\operatorname{Ad}(x)\mathfrak{l} \subseteq \mathfrak{l}$ and $x \in L$. If instead $x \in L$, then $\operatorname{Ad}(x)\mathfrak{l} \subseteq \mathfrak{l} \subseteq \mathfrak{q}$ and $\operatorname{Ad}(x)\mathfrak{l} \subseteq \mathfrak{l} \subseteq \theta\mathfrak{q}$ by definition. Hence, $x \in Q \cap \Theta Q$. This completes the proof. □

Combining this Proposition with the discussion above, we see that

$$L = N_Q(\mathfrak{l}) = Z_G(\mathfrak{h}'') = Q \cap \Theta Q$$

Theorem 2.1.10. *Every parabolic subalgebra $\mathfrak{q}$ containing the Borel subalgebra $\mathfrak{b}$ can be decomposed into a vector space direct sum $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ such that $\mathfrak{m}$ is reductive, $\mathfrak{a}$ is abelian, and $\mathfrak{n}$ is nilpotent.*

Proof. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a Levi decomposition. By Proposition 2.1.5, we have that $\mathfrak{l} = \mathfrak{h}'' \oplus \mathfrak{l}_{ss}$. Then $\mathfrak{q} = \mathfrak{h}'' \oplus \mathfrak{l}_{ss} \oplus \mathfrak{u}$. Put

$$\mathfrak{m} = \{X + \theta X : X \in \mathfrak{g}_\alpha, \alpha \in \Gamma \cap -\Gamma\} \oplus \mathfrak{h}'$$

Then $\mathfrak{m}$ is reductive as it is θ-stable. Further, put $\mathfrak{n} = \mathfrak{u} \oplus \bigoplus_{\alpha \in \Gamma, \alpha > 0} \mathfrak{g}_\alpha$. Now,

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{h}'' \oplus \mathfrak{n}$$

by simply noticing that for any $\alpha \in \Gamma \cap -\Gamma, \alpha < 0$, $\mathfrak{g}_\alpha \subseteq \mathfrak{m} \oplus \mathfrak{n}$ by writing any element $X \in \mathfrak{g}_\alpha$ as $X + \theta(X) - \theta(X)$. By writing $\mathfrak{a} = \mathfrak{h}''$, we have that $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and this completes the proof. ☐

Remark 2.1.11. The decomposition of $\mathfrak{q}$ above is called the **Langlands Decomposition**. Once we give the associated decomposition for real Lie algebras, we can begin to understand the representation theory of non-compact Lie groups by looking at those representations which are induced from the parabolics. This will mirror the Borel-Weil theorem, but will be a significant generalization.

The final goal of this section is to begin to investigate the significance of parabolic subalgebras in the finite dimensional representation theory of $\mathfrak{g}$. What we shall see is that the finite dimensional representation theory of the Levi factor is incredibly similar to the finite dimensional representation theory of $\mathfrak{g}$. In the infinite dimensional case, the representation theory of parabolic subalgebras (and subgroups) plays an essential role in classifying the irreducible representations of $\mathfrak{g}$ and G. Before we get to the main result, we need a technical proposition.

Proposition 2.1.12. *Let $\mathfrak{q}$ be a parabolic subalgebra containing $\mathfrak{b}$ and $\mathfrak{l} \oplus \mathfrak{u}$ a Levi decomposition. Then in any representation of $\mathfrak{l}$ where $\mathfrak{h}$ acts completely reducibly, then $\mathfrak{l}$ acts completely reducibly. In particular for all $\mathfrak{g}$-representations, $\mathfrak{l}$ acts irreducibly.*

Proof. Let (π, V) be any representation of $\mathfrak{l}$. If $\mathfrak{h}$ acts completely reducibly, then $\mathfrak{h}''$ acts completely reducibly. By considering V as a $\mathfrak{l}_{ss}$-module, we see that V decomposes as a direct sum of irreducible $\mathfrak{l}_{ss}$-modules. As $\mathfrak{l} = \mathfrak{h}'' \oplus \mathfrak{l}_{ss}$, each simultaneous eigenspace of $\mathfrak{h}''$ will induce a representation of $\mathfrak{l}_{ss}$ and hence $\mathfrak{l}$. This completes the proof. ☐

In the case of $\mathfrak{g}$-representations, we want to know how much information we can extract from $\mathfrak{l}$. The answer will turn out to be a surprising amount. Before we get to that though, we need to define a particular subspace on which $\mathfrak{l}$ will act naturally.

Definition 2.1.13. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a Levi decomposition of a parabolic subalgebra containing the standard Borel subalgebra. Let (π, V) be a $\mathfrak{g}$-representation. The space of $\mathfrak{u}$-**invariants** of V is the space $V^{\mathfrak{u}}$ which is defined as

$$V^{\mathfrak{u}} = \{v \in V : \pi(X)v = 0, \forall X \in \mathfrak{u}\}$$

This is a representation of $\mathfrak{l}$ as $\mathfrak{u}$ is an ideal in $\mathfrak{q}$ and hence

$$X(Lv) = L(Xv) + [X, L]v = 0$$

The representation $V^{\mathfrak{u}}$ is determined up to equivalence by the representation of $\mathfrak{h}$ on the space of $\mathfrak{l} \cap \mathfrak{n}$ invariants. Now,

$$(V^{\mathfrak{u}})^{\mathfrak{l}\cap\mathfrak{n}} = V^{\mathfrak{u}\oplus(\mathfrak{l}\cap\mathfrak{n})} = V^{\mathfrak{n}} \tag{2.5}$$

and the right side is given by the Theorem of Highest Weights for $\mathfrak{g}$.. Therefore, the representation $V^{\mathfrak{u}}$ is a generalization of the highest weight representation of $\mathfrak{g}$ on V. The following theorem formalizes this relationship.

Theorem 2.1.14 (Reduction to Parabolics)**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{q}$ be a parabolic subalgebra containing the Borel subalgebra.*

(a) If (π, V) is an irreducible finite dimensional representation of $\mathfrak{g}$, then $(\pi, V^{\mathfrak{u}})$ is an irreducible finite dimensional representation of $\mathfrak{l}$ with the same highest weight.

(b) If V_1 and V_2 are two irreducible finite dimensional representations of $\mathfrak{g}$ so that $V_1^{\mathfrak{u}}$ and $V_2^{\mathfrak{u}}$ are equivalent representations of $\mathfrak{l}$, then V_1 is equivalent to V_2 as $\mathfrak{g}$-representations.

(c) If an irreducible finite dimensional representation M of $\mathfrak{l}$ has highest weight dominant and algebraically integral, then there exists a $\mathfrak{g}$-representation V so that $M \cong V^{\mathfrak{u}}$.

Proof. For (a), we invoke the Theorem of Highest weights to see that $V^{\mathfrak{n}}$ is 1-dimensional. Therefore, the space of $\mathfrak{l} \cap \mathfrak{n}$ invariants of $V^{\mathfrak{u}}$ is 1-dimensional by (2.4). Since $V^{\mathfrak{u}}$ is completely reducible as an $\mathfrak{l}$-representation, the Theorem of Highest weights shows that it is irreducible. If λ is the highest weight of V under $\mathfrak{g}$, then λ is the highest weight of $V^{\mathfrak{u}}$ since $V_\lambda = V^{\mathfrak{n}} \subseteq V^{\mathfrak{u}}$. Hence, λ is dominant and algebraically integral.

For (b), we simply apply uniqueness from the Theorem of Highest Weights. If $V_1^{\mathfrak{u}}$ is equivalent to $V_2^{\mathfrak{u}}$ then by taking $\mathfrak{l} \cap \mathfrak{u}$ invariants, we see that these. spaces carry the same representation of $\mathfrak{g}$. Hence, V_1 and V_2 are equivalent as $\mathfrak{g}$ representations.

For (c), let M have highest weight λ. Then if V is the associated highest weight representation of $\mathfrak{g}$ with highest weight λ, then $V^{\mathfrak{u}} \cong M$ by the uniqueness is the theorem of highest weights and part (a). This completes the proof. □

Using this theorem, we can now decompose (π, V) into $V^{\mathfrak{u}}$ and its orthogonal complement. The following proposition will give us a concrete characterization of this orthogonal complement, as well as precisely stating how the $\mathfrak{I}$ (and thus $U(\mathfrak{l})$) structure on $V^{\mathfrak{u}}$ determines the $\mathfrak{g}$ representation.

Proposition 2.1.15. *Let V be any finite dimensional $U(\mathfrak{g})$ module and $\mathfrak{q}$ a parabolic subalgebra of $\mathfrak{g}$. Then:*

(a) $V = V^{\mathfrak{u}} \oplus \bar{\mathfrak{u}}V$

(b) The natural map $V^{\mathfrak{u}} \to V/(\bar{\mathfrak{u}}V)$ is an isomorphism of $U(\mathfrak{l})$ modules.

(c) The $U(\mathfrak{l})$ module $V^{\mathfrak{u}}$ determines the $U(\mathfrak{g})$ module V up to equivalence; the number of irreducible constitutents of $V^{\mathfrak{u}}$ equials number of irreducible constituents of V, and the multiplicities of these irreducible components are the same as for V.

Proof. It is clear that $V^{\mathfrak{u}}$ and $\bar{\mathfrak{u}}V$ are $\mathfrak{l}$-representations and thus $U(\mathfrak{l})$-modules. (b) follows from (a) immediately, and (c) follows from the previous theorem. Thus is suffices to prove (a).

Without loss of generality, assume V is irreducible with highest weight λ. Let $E_{-\alpha}, H_\alpha, E_\alpha$ be a Poincare-Birkoff-Witt Basis for $U(\mathfrak{g})$. As $V^{\mathfrak{u}}$ is annihilated by all elements of the form $E_\alpha, \alpha \in \Gamma, \alpha \notin -\Gamma$. Therefore, we have a decomposition of V as a $U(\mathfrak{g})$-module

$$V = U(\mathfrak{l})V_\lambda \oplus \bar{\mathfrak{u}}V$$

As $\mathfrak{l}$ acts irreducibly on $V^{\mathfrak{u}}$ by the preceding theorem and $V_\lambda = V^{\mathfrak{n}} \subseteq V^{\mathfrak{u}}$, we see that $U(\mathfrak{l})V_\lambda = V^{\mathfrak{u}}$. This proves (a) and completes the proof of the proposition. ☐

2.1.2 Parabolic Subalgebras: the real (reductive) case

Now we turn our attention to $\mathfrak{g}_0$ and will attempt to mirror the above constructions in the real case. The main difference here is that we need to be careful when discussing Cartan subalgebras which are no longer conjugate in $\mathfrak{g}_0$. In particular, our parabolics will depend on the choice of Cartan decomposition. This is because we are considering the set of restricted roots $\Sigma(\mathfrak{g}_0, \mathfrak{a})$ and thus the choice of minimal parabolic (the analogue to the Borel subalgebra) comes from the Cartan decomposition of $\mathfrak{g}_0$. For ease of notation, we shall denote the lie algebras in this section by $\mathfrak{g}$ and their complexification as $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{g}$ be a real reductive Lie algebra and θ a Cartan involution. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition and $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$. Put $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be restricted roots and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the associated Iwasawa decomposition. If $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, then $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra of $\mathfrak{g}$. Pick a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{m}$. Then $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ and by construction it is θ-stable.

Definition 2.1.16. We say that a subalgebra $\mathfrak{b}$ is a **minimal parabolic subalgebra** if $\mathfrak{b}$ is conjugate (necessarily via an inner automorphism as reductive groups are in the Harish-Chandra class) to $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Any subalgebra containing such a minimal parabolic is called a **parabolic subalgebra**.

One question that may arise here is the discrepancy between the inclusion of $\mathfrak{m}$ in the real case versus the complex one. If we were to mirror the complex case naively we would only want to include $\mathfrak{t} \oplus \mathfrak{a}$ as this is a Cartan subalgebra for $\mathfrak{g}$. However, it is not the fact that $\mathfrak{h}$ is a Cartan subalgebra that is important here. The key is that $\mathfrak{h}$ in the complex case is the 0-root space. In the real situation, this is mirrored precisely as $\mathfrak{m} \oplus \mathfrak{a}$ is the 0-restricted root space and thus plays the same role as the Cartan subalgebra in the complex version of the. decomposition.

Remark 2.1.17. Equivalently, we could have used our discussion from above to define parabolic subalgebras in the real setting as subalgebras $\mathfrak{s}$ for which $\mathfrak{s}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. As it turns out, these definitions are equivalent. We shall not prove this here.

Similar to the complex case, we can immediately build a family of parabolic subalgebras containing $\mathfrak{b}$: let $\Pi' \subseteq \Pi$ be a subset of the simple restricted roots, and put

$$\Gamma_{\Pi'} = \Sigma^+ \cup \mathrm{Span}_{\mathbb{Z}}(\Pi').$$

Then the parabolic subalgebra associated to Π' is defined as

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Gamma_{\Pi'}} \mathfrak{g}_\alpha$$

By inspecting the proof of Proposition 2.1.3, we see that nowhere do we use the fact that $\mathfrak{q}$ is defined over $\mathbb{C}$ and thus can apply this result here to conclude the following

Proposition 2.1.18. *The parabolic subalgebras $\mathfrak{q}$ containing $\mathfrak{b}$ are parametrized by the set of subsets of simple roots. The correspondence is given by $\Pi' \mapsto \mathfrak{q}(\Pi') = \mathfrak{m} \oplus \mathfrak{a} \bigoplus_{\Gamma_{\Pi'}} \mathfrak{g}_\alpha$.*

From this we again deduce the Levi decomposition of $\mathfrak{q}$ with

$$\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha \qquad\qquad \mathfrak{u} = \bigoplus_{\substack{\alpha \in \Gamma \\ \alpha \notin -\Gamma}} \mathfrak{g}_\alpha$$

By construction $\mathfrak{l}$ is θ-stable, hence reductive, and $\mathfrak{u}$ is an ideal.

Theorem 2.1.19 (Langlands Decomposition of the Lie algebra)**.** *Let $\mathfrak{q}$ be a parabolic subalgebra containing $\mathfrak{b}$. Then there exist Lie subalgebras $\mathfrak{m}_0, \mathfrak{a}, \mathfrak{n}$ so that $\mathfrak{q} = \mathfrak{m}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.*

Proof. Put $\mathfrak{a}' = \bigcap_{\alpha \in \Gamma \cap -\Gamma} \ker \alpha \subseteq \mathfrak{a}$. This is clearly abelian and central in $\mathfrak{l}$. Further, by the argument for (2.3) we see that $\mathfrak{l} = Z_{\mathfrak{g}}(\mathfrak{a}')$. Let $\mathfrak{a}'^{\perp}$ be the orthogonal complement to $\mathfrak{a}'$ in $\mathfrak{a}$. As $\mathfrak{a}$ is abelian, we see that $\mathfrak{a}'^{\perp} \subseteq \mathfrak{l}$. Further, $\mathfrak{m} \subseteq \mathfrak{l}$ by definition. Thus, if

$$\mathfrak{m}_0 = \mathfrak{m} \oplus \mathfrak{a}'^{\perp} \oplus \bigoplus_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha$$

we see that $\mathfrak{l} = \mathfrak{m}_0 \oplus \mathfrak{a}'$. By the Levi decomposition, we see that

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u} = \mathfrak{m}_0 \oplus \mathfrak{a}' \oplus \mathfrak{u}$$

Finally, $\mathfrak{m}_0$ is θ-stable as each summand is θ-stable. Hence, we have found the desired decomposition. $\square$

Continuing as in the previous section define the following groups:

$$\begin{array}{lll} Q = N_G(\mathfrak{q}) & L = Q \cap \Theta Q = Z_G(\mathfrak{a}') & M = (L \cap K) \exp(\mathfrak{a}'^{\perp} \oplus \mathfrak{l} \cap \mathfrak{n}) \\ A' = \exp \mathfrak{a}' & & U = \exp \mathfrak{u} \end{array}$$

Theorem 2.1.20 (Langlands Decomposition of the Lie Group)**.** *In the notation above, we have that following:*

(a) The multiplication map $M \times A' \times U \to Q$ is a diffeomorphism.

(b) MA' normalizes U.

(c) $\Theta U \cap Q = \{1\}$

Proof. For (a), notice that by Lemma 1.2.15 we have that the map $M \times A' \times U \to Q$ is everywhere regular. It remains to prove that it is injective. To do this, consider an Iwasawa decomposition of $M = K_M A_M N_M$. Then it is clear that if $G = KAN$ is an Iwasawa decomposition, then $A = A_M A'$ and $N = N_M U$. Now, injectiviity of the multiplication map is equivalent to showing that $M \cap A'U = \{1\}$. To do this, let $X \in \mathfrak{a}' \oplus \mathfrak{u}$ such that $\exp X \in M$. Then X centralizes $\mathfrak{a}'$ and hence, $X \in \mathfrak{m}_0 \cap \mathfrak{a}' \oplus \mathfrak{u} = \{0\}$. Hence, X is 0 and $\exp X = 1$. Therefore, the map is a diffeomorphism.

(b) and (c) follow from the definitions on the Lie algebra level and arguing by contradiction. $\square$

There is a particular case of parabolic subgroups which are of such interest, they have a special name.

Definition 2.1.21. Let $Q = MA'U$ be a parabolic subgroup. We say that Q is **cuspidal** if $\mathfrak{m}_0$ contains a θ-stable, compact Cartan subalgebra $\mathfrak{t}$. In particular, $\mathfrak{t} \subseteq \mathfrak{m}$. We shall not go into any detail of cuspidal parabolics but instead suggest [Kna05] for a full account. As it turns out, the cuspidal parabolic subgroups are particularly easy to work with because the Langlands decomposition arises naturally from the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to the θ-stable Cartan subalgebra $\mathfrak{h} = (\mathfrak{t} \oplus \mathfrak{a}')_{\mathbb{C}}$.

2.2 Infinite-dimensional representations

We can now begin to tackle infinite dimensional representations of G. As it will turn out, most of the irreducible representations of a non-compact Lie group are infinite dimensional. This introduces many analytic obstructions to mimicking the compact story. It was the great insight of Harish-Chandra and later Lepowsky, that for reductive Lie groups there is a certain functor which can be applied to any complex representation (π, V) of G to produce a so-called $(\mathfrak{g}_{\mathbb{C}}, K)$-module (π, V_K). This infinitesimal version of the global representation (infinitesimal here is with respect to the Lie algebra) will turn out to house nearly all of the information about (π, V) and it comes with the great advantage: it is purely algebraic. For this reason, $(\mathfrak{g}_{\mathbb{C}}, K)$-module theory has become an essential tool in understanding representations and (as well shall see in the next chapter) it is. the correct setting for the Langlands classification. Broadly, this classifies all $(\mathfrak{g}_{\mathbb{C}}, K)$-modules of a given type.

Before we can get into this theory, we first need to continue where we left off: that is with parabolic subgroups. As a heuristic for the future, when we want to understand representations of G, there are two ways of doing so: (1) *restriction* to a subgroup (this could be any group: K, Q, M, etc.) or (2) *induction* where by we start with a representation of a subgroup H and enlarge it to G. The goal of the next subsection is to write this story down in the case where G is compact.

Remark 2.2.1. A note to someone reading this: you may wonder why we care about compact groups in this setting. G for instance is non-compact and thus it would seem on first glance that the compact picture should be irrelevant: this is wrong. Similar to how the Cartan decomposition told us that G and K have the same homotopy type, we can further this result (using induction and restriction) to show that the representation theory of G is highly controlled by restricting these representations to K. So controlled in fact, that when we want to classify all representations of a given type, we shall find that we *only* need to inspect K and the closely related subgroup $K \cap M$. The first hint that this is the correct approach is the combination of the Langlands and Iwasawa decompositions for G. If we were to induce a representation from $MA'U \to KAN$, we see will see that the induction mainly acts on $M \cap K$ and K.

2.2.1 Frobenius Reciprocity and Compact Induction

For this section, let K be a compact Lie group and M a subgroup. In the case that K and M are finite, we have the well known result under the name *Frobenius reciprocity* which gives an adjoint pair of functors

$$\mathrm{Ind}_{K\cap M}^{K} : K \cap M - \mathbf{Mod} \leftrightarrows K - \mathbf{Mod} : \mathrm{Res}_{K\cap M}^{K}$$

The goal of this section is to prove Frobenius reciprocity in the more general setting above. Once complication that will arise fairly naturally is that because K and M have differentiable (and analytic) structures, we can induce representations and only look at the continuous, differentiable, L^2, L^p etc. class of functions inside there. This causes some small technical points, but is dealt with quite nicely in this compact case. As we shall see in Section 2.4, we care about such a construction because we want to understand induced representations from Parabolic subgroups. By the Langlands and Iwasawa decompositions, we see that this boils down to understanding the induced representation of $K \cap M$ to K.

The story here begins with some generalities about representation theory of K on a complex Hilbert space $\mathcal{H}$ (of possibly infinite dimension). By writing $\langle , \rangle$ for the inner product on $\mathcal{H}$, we have the associated sublinear functional $|| \cdot ||$ give by $||h|| = \sqrt{\langle h, h\rangle}$ for any $h \in \mathcal{H}$.

Definition 2.2.2. We say that a linear operator T (linear endomorphism) is **unitary** if $||Tv|| = ||v||$ for all $v \in \mathcal{H}$. In this way, $T^{-1} = T^*$, the adjoint, and the set of unitary operators forms a group denoted by $U(\mathcal{H})$. This is called the **unitary group** of $\mathcal{H}$.

The most important Hilbert space for our purposes (and for the purposes of Harmonic analyists and Harish-Chandra) will be $L^2(K)$ (and more generally $L^2(G)$). On this space, there are two natural actions given by the following formulas:

$$L(g)(f)(x) = f(g^{-1}x) \qquad\qquad R(g)(f)(x) = f(xg)$$

these are called the **left regular** and **right regular** representations of K on $L^2(K)$. In fact, if F is any space of functions whose source space is K, then the left and right regular representations are defined there. It is a well known fact that these two representations commute with one another and is easily checked that for each $g \in K$ both $L(g)$ and $R(g)$ are unitary operators on $L^2(K)$.

Definition 2.2.3. By a **unitary representation** K on $\mathcal{H}$, we mean a group homomorphism $K \to U(\mathcal{H})$ such that the map

$$K \times \mathcal{H} \to \mathcal{H} \qquad\qquad (k, v) \mapsto \Pi(k)v$$

is continuous. We say that two unitary representations $(\pi, V), (\sigma, U)$ are **unitarily equivalent** if there exists a norm-preserving linear map $T : V \to U$ with a norm-preserving inverse such that $\pi(k)T = T\sigma(k)$ for all $k \in K$.

Lemma 2.2.4. *Let $\Pi : K \to U(\mathcal{H})$ be a group homomorphism. Then Π is a unitary representation if and only if the map $k \mapsto \Pi(k)v$ is continuous for all $v \in \mathcal{H}$.*

Remark 2.2.5. It will sometimes be of great use to consider an extension of the representation to $C(K), C^\infty(K)$, and $L^1(K)$ (when G is a non-compact group, we extend to $C_c^\infty(G)$ the compactly supported functions under convolution). To do so, put

$$\Pi(f)v = \int_K f(x)\Pi(x)v dx$$

It is then (somewhat tediously) checked that $\Pi(f * g) = \Pi(f)\Pi(g)$ and thus Π extends to a representation of the Banach algebras above.

By the Peter-Weyl theorem, all of the irreducible representations of K are finite dimensional and can be taken to be unitary. What we would like to know is that $\mathcal{H}$ decomposes into irreducible representations. As we are in the infinite dimensional domain, we need to not only consider algebraic decompositions, but also analytic ones. The following proposition simplifies our situation:

Proposition 2.2.6. *Every unitary representation of K decomposes as a Hilbert space direct sum of irreducible finite dimensional representations of K.*

Definition 2.2.7. Let $\widehat{K}$ denote the collection of all equivalence classes of irreducible representations of K. Let $(\Pi, \mathcal{H})$ be any unitary Hilbert space representation of K, then for any $\gamma \in \widehat{K}$ denote by

$$m_\gamma(\mathcal{H}) := \dim_{\mathbb{C}} \mathrm{Hom}(V_\gamma, \mathcal{H})$$

This is the **multiplicity** of γ in $\mathcal{H}$.

As it will turn out, the multiplicities of all irreducible representations contained in the decomposition of $\mathcal{H}$ are a fairly strong invariant and thus can be used to separate inequivalent representations. Furthermore, multiplicities interact well with the restriction and induction operators hinted at above. The precise formulation of this statement will be seen shortly.

Before then, we need to generalize the induced representations of the Borel-Weil theorem. In the context of that theorem we take an analytically integral, dominant weight λ, construct a homogeneous line bundle L_λ over the complex flag manifold K/T and build an irreducible representation out of holomorphic sections of this bundle. The assumption of holomorphicity is to ensure that $\dim_{\mathbb{C}} \Gamma_{Hol}(K/T, L_\lambda) < \infty$. If we remove this restriction, we immediately get an infinite dimensional representation, and call also extend our notions away from the complex domain. To generalize this correctly, we proceed in the following way: let $T \leq K$ be a closed subgroup. Then K/T has the structure of a manifold, and any Hilbert space representation (σ, V) of T creates a homogenous vector bundle

$$W_\sigma := K \times_T V \to K/T$$

Definition 2.2.8. Denote by $(\mathrm{Ind}_T^K(\sigma), C^0(K/T, W_\sigma))$ the representation of K on the space of continuous sections of W_σ. This is called the **continuous induced representation** from T to K. If instead we wanted to look at smooth sections, or even L^2 sections, we can similarly do this. The L^2-case requires a bit of comment. Let V a Hilbert space. A function $F : K \to V$ is measurable if $x \mapsto \langle F(x), v\rangle$ is Borel measurable. Then if $\{v_n\}$ is an orthonormal basis of V, we put $|F(x)|^2 = \sum |\langle F(x), v_n\rangle|^2$. Put $L^2(K, V)$ to be the set of all functions $F : K \to V$ which are measurable and if $||F||_2 = \left(\int_K |F(x)|^2 dx\right)^{1/2}$. Denote by

$$L^2(K, V, \sigma) = \{f \in L^2(K, V) \,|\, f(kt) = \sigma(t)^{-1} f(k) \text{ for almost every pair } (k, t) \in K \times T\}$$

These are the L^2-sections of the bundle. Defining these functions almost everywhere indicates that we may have some issue with evaluation at given points, but this can be avoided in nearly all instances.

Remark 2.2.9. For the purposes of this section, we shall always consider L^2 sections of W_σ.

The following theorem is a more general version of Frobenius reciprocity from the character theory on finite groups.

Theorem 2.2.10 (Frobenius Reciprocity)**.** *Let $L \leq K$ be a closed subgroup, (σ, V_σ) any irreducible unitary representation of L, and (τ, V_τ) any irreducible unitary representation of G. Then we have the following natural isomorphism*

$$\mathrm{Hom}_K(V_\tau, L^2(K, V_\sigma, \sigma)) \cong \mathrm{Hom}_L(V_\tau, V_\sigma)$$

and the following equality of multiplicities

$$m_\tau(\mathrm{Ind}_L^K(\sigma)) = m_\sigma(\tau|_L)$$

Proof. Observe first that $L^2(K, V, \sigma)$ is the direct sum of $d_\sigma := \dim V_\sigma$ copies of $L^2(K)$. Therefore, we see that τ occurs exactly $d_\sigma d_\tau$ times in $L^2(K, V_\sigma)$ and at most this many times in $L^2(K, V_\sigma, \sigma)$. By Schur's Lemma, any member of $\mathrm{Hom}(V_\tau, L^2(K, V_\sigma, \sigma))$ consists of continuous function and thus, can evaluate the images in a well-defined fashion. Now consider the "evaluation at 1" morphism ev_1. Then for every $A \in \mathrm{Hom}(V_\tau, L^2(K, V_\sigma, \sigma))$, we can take the composite map

$$\mathrm{ev}_1 \circ A : V_\tau \to V$$

By restricting the domain, we get an element in $\mathrm{Hom}_L(V_\tau, V_\sigma)$. It remains to show that the assignment

$$A \mapsto A(-)(1)$$

is bijective.

To show it is injective, suppose $A \in \ker \mathrm{ev}_1$. Then $(Av)(1) = 0$ for all $v \in V_\tau$. Applying this conclusion to an element $\tau(k)^{-1}v'$ we see that

$$0 = (A\tau(k)^{-1}v')(1) = (\mathrm{Ind}_L^K(\sigma)(k)Av')(1) = Av'(k)$$

by definition. Hence, $Av' \equiv 0$. Since, v' is arbitrary, we see that $A = 0$ and ev_1 is injective.

To show it is surjective, let $a \in \mathrm{Hom}_L(V_\tau, V_\sigma)$. Put $Av(k) = a(\tau(k)^{-1}v)$ for all $v \in V_\tau, k \in K$. Then by construction

$$Av(gh) = \sigma(h)^{-1}(Av)(g)$$

and thus $Av \in L^2(K, V_\sigma, \sigma)$. It remains to show that $A \in \mathrm{Hom}_K(V_\tau, L^2(K, V_\sigma, \sigma))$. This follows though, from the equality

$$(\mathrm{Ind}_L^K(\sigma)(k)Av)(k') = a(\tau(k')^{-1}(\tau(k)v)) = A(\tau(k)v)(g)$$

Hence, $\mathrm{Ind}_L^K(\sigma)(k)A = A\tau(k)$ for all $k \in K$. Lastly, it is obvious that $A \mapsto a$ and hence ev_1 is a bijection. The formula for multiplicities then follows immediately. $\square$

In particular, this result tells us that $\mathrm{Res}^K_L \dashv \mathrm{Ind}^K_L$ as functors between the respective categories of modules. It should be clear that $\mathrm{Res}^K_L \mathrm{Res}^L_{L'} = \mathrm{Res}^K_{L'}$ as functors. The story for Ind is not too different, but a small technical point must be made.

Proposition 2.2.11 (Induction in Stages)**.** *Let K be a compact Lie group and L, L_1 closed subgroups such that $L \subseteq L_1 \subseteq K$. If σ is an irreducible unitary representation of L, then*

$$\mathrm{Ind}^K_L(\sigma) \textit{ is unitarily equivalent with } \mathrm{Ind}^K_{L_1}\mathrm{Ind}^{L_1}_L(\sigma)$$

Proof. Notice that for any unitary representation τ of K, we have the following formula for multiplicities:

$$m_\sigma(\tau) = \sum_{\gamma \in \widehat{L_1}} m_\gamma(\tau) m_\sigma(\gamma)$$

Frobenius reciprocity, then gives

$$m_\tau(\mathrm{Ind}^K_L(\sigma)) = \sum_{\gamma \in L_1} m_\tau(\mathrm{Ind}^K_{L_1}(\gamma)) m_\gamma(\mathrm{Ind}^{L_1}_L(\sigma))$$

The representation $\mathrm{Ind}^{L_1}_L(\sigma)$ is an orthogonal direct sum over all $\gamma \in \widehat{L_1}$ with a multiplicity number of copies, and hence is unitarily equivalent to the orthogonal direct sum over all $\gamma \in \widehat{L_1}$ of $m_\gamma(\mathrm{Ind}^{H_1}_H(\sigma))$ copies of $\mathrm{Ind}^G_{H_1}(\gamma)$. Thus, the right side of the formula above is

$$m_\tau(\mathrm{Ind}^K_{L_1}\mathrm{Ind}^{L_1}_L(\sigma))$$

Hence, the representations have the same multiplicities and thus are unitarily equivalent. □

Notice that we do not get equality on the nose unlike in the case of restrictions. For this reason, there is a question as to which induction we should use. One may wonder if instead of L^2 induction we do C^∞ or just continuous induction, we may get equality. As it will turn out, the answer is still no. What will be particularly interesting however, is that to each of these representations we can associate an infinitesimal (or well-behaved in the language of Harish-Chandra) representation and these representations will turn out to be the same! To make some sense of this, and to fully pass to the world of $(\mathfrak{g}_\mathbb{C}, K)$-modules, we need to understand how to build these infinitesimal representations and in particular, how to define an action of the Lie algebra on an infinite dimensional representation of G.

2.2.2 Smooth and K-finite vectors

We now return to the general case. What we present here again will be true for groups in the Harish-Chandra class, but the proofs will be presented for reductive groups unless noted otherwise. Let G be a real reductive Lie group, $\mathfrak{g}_0$ its Lie algebra, and $\mathfrak{g}$ its complexification. Let Θ be a global Cartan involution and K the maximal compact subgroup corresponding to the choice of Θ. We have the following decompositions from chapter 1:

$G = K \times \mathfrak{p}_0$	Polar Decomposition
$G = KAN$	Iwasawa Decomposition
$G = KAK$	Cartan Decomposition

Let (Π, V) be any infinite dimensional complex representation of G on a Hilbert (or even Banach, Fréchet, etc.) space V with inner product $\langle , \rangle$. Unlike the finite dimensional case, the following limit may not exist:

$$\lim_{t\to 0} \frac{\Pi(\exp tX)v - v}{t} \tag{2.6}$$

for $g \in G$ and $v \in V$.

Definition 2.2.12. We say that a vector $v \in V$ is of **class** C^1 if the mapping $v \mapsto \lim_{t\to 0} \frac{\Pi(\exp tX)v - v}{t}$ is continuous. We call a vector **smooth** or of class C^∞ if the mapping

$$v \mapsto \Pi(g)v$$

is C^k for all $k \geq 0$. Let $V^\infty \subseteq V$ denote the subspace of all smooth vectors.

Lemma 2.2.13. (Π, V^∞) *is a representation of* G*. Further, we can differentiate this action to get a representation* (π, V^∞) *of* $\mathfrak{g}$*.*

Proof. Let $v \in V^\infty$. Then for all $g, h \in G$, $gh \mapsto \Pi(gh)v$ is C^∞. As Π is a homomorphism, $\Pi(g)\Pi(h)v \in V^\infty$ and by associativity

$$\Pi(gh)v = \Pi(g) \cdot (\Pi(h)v) \in V^\infty$$

Thus, $\Pi(h)v \in V^\infty$. As $g, h \in G$ are arbitrary, we see that $\Pi(g)v \in V^\infty$ for all $v \in V^\infty$ and (Π, V^∞) is a (smooth) representation of G.

To define the Lie algebra representation, let $X \in \mathfrak{g}_0$ and for each $v \in V^\infty$ put

$$\pi(X)v = \lim_{t\to 0} \frac{\Pi(\exp tX)v - v}{t} = \frac{d}{dt}\Pi(\exp tX)v|_{t=0}$$

This is well-defined as $v \in V^\infty$ and also leaves V^∞ invariant as $\Pi(\exp tX)v$ is smooth. It remains to show that π is a Lie algebra homomorphism. (Note that if V is finite dimensional then every vector is smooth and π is a Lie algebra homomorphism as it is the tangent map to a Lie group homomorphism)

Let $X \in \mathfrak{g}_0$ and put $\tilde{X}$ the corresponding left-invariant vector field on G. Then if we put $f_v(g) = \Pi(g)v$, we see that

$$\Pi(g)\pi(X)v = (\tilde{X}f_v)(g)$$

By putting $g = \exp tY$ for $Y \in \mathfrak{g}_0$, we obtain

$$\pi(Y)\pi(X)v = \frac{d}{dt}\Big|_{t=0}\Pi(\exp tY)\pi(X)v = \frac{d}{dt}\Big|_{t=0}(\tilde{X}f_v)(\exp tY) = \tilde{Y}(\tilde{X}f_v)(e)$$

Interchanging Y and X and subtracting them, we get

$$\pi(X)\pi(Y)v - \pi(Y)\pi(X)v = (\tilde{X}(\tilde{Y}f_v) - \tilde{Y}(\tilde{X}f_v))(e)$$

The right hand side is precisely $([\tilde{X}, \tilde{Y}]f_v)(e) = \widetilde{[X, Y]}f_v(e)$ and thus π is a Lie algebra representation. As V^∞ is a complex vector space, we see that π extends to a complex Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(V^\infty)$ and thus an associative algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}(V^\infty)$ which sends 1 to 1. $\square$

One key aspect of this procedure is that V^∞ is not some arbitrary subspace of V as the following example shows:

Example 2.2.14. Let $G = \mathbb{R}$ and $(\lambda, L^2(\mathbb{R}))$ be the left-regular representation of G. Put $V = \{f \in C_c^\infty(\mathbb{R}) : \operatorname{supp} f \subseteq [0,1]\}$. Then V is $\mathfrak{g}$ invariant, but neither V, nor its closure are G invariant.

What we want to show is that V^∞ is not only a suitably nice subspace, but that $V = \overline{V^\infty}$. To do this, first will extend our given representation Π from G to $C_c^\infty(G)$. From here, it will then follow that the set of all matrix coefficients of this extension will be dense in V and will all be smooth. We start with the extension.

Definition 2.2.15. Let (Π, V) be a Hilbert space representation of G. For any $f \in C_c^\infty(G)$, put

$$\Pi(f)v = \int_G f(g)\Pi(g)v \, dg$$

where dg denotes a left-Haar measure on G. Notice that $||\Pi(f)v|| \leq C_\Omega ||v|| ||f||_1$ for all compact $\Omega \subseteq G$, where C_Ω is a constant. Define the **Gårding Subspace** of V to be the linear span of the $\Pi(f)v$.

The reason we consider such a space is that we can chose a sequence of f_i to be an *approximation to the identity* (a net of distributions which converges to the identity element). This in turn tells us that at least some of the $\Pi(f)v$ are smooth vectors. As the following proposition shows, in fact all of the $\Pi(f)v$ are smooth. Even more convenient, the G̊ding subspace is dense in V and by extension V^∞ is dense.

Proposition 2.2.16. *Let (Π, V) be a representation of G and $f \in C_c^\infty(G)$. Then the following are true:*

(a) For every $v \in V$, $\Pi(f)v \in V^\infty$.

(b) The Gårding subspace of V is dense in V.

(c) V^∞ is dense in V.

Proof. For (a), we first show that for $X \in \mathfrak{g}$ then $\pi(X)\Pi(f)v$ exists. By the definitions:

$$\pi(X)\Pi(f)v = t^{-1} \int_G f(g) \left(\Pi(\exp tX)\Pi(g) - \Pi(g)\right) \, dg$$

By making the substitution $g \mapsto \exp(-tX)g$, we get that

$$t^{-1} \int_G f(g) \left(\Pi(\exp tX)\Pi(g) - \Pi(g)\right) \, dg = \int_G \frac{f(\exp(-tX)g) - f(g)}{t} \Pi(g)v \, dg$$

By taking the limit as $t \to 0$ and applying the Dominated Convergence Theorem, we see that $\pi(X)\Pi(f)v = -\Pi(Xf)v$ and thus exists. This shows that the Gårding subspace is stable under $\mathfrak{g}$ and thus consists of smooth vectors.

For (b), let $v \in V$. By assumption Π is continuous and thus for any $\varepsilon > 0$ the set

$$T = \{g \in G : ||\Pi(g)v - v|| < \varepsilon\}$$

is open. Therefore, there exists some $C \subseteq T$ compact and $f \in C_c^\infty(G)$ with $\operatorname{supp} f \subseteq C$. By normalizing, we can assume $\int_G f\, dg = 1$ and thus

$$||\Pi(f)v - v|| = ||\int_G f(g)[\Pi(g)v - v]\, dg|| \leq \int_G f(g)||\Pi(g)v - v||\, dg \leq \varepsilon \int_G f(g)\, dg = \varepsilon$$

Hence, the Gårding subspace is dense in V. Part (c) follows from (a) and (b). This completes the proof. □

We now want to mirror the finite-dimensional case. The main obstruction to directly doing this being that characters no longer entirely determine the representation (namely because they do not exist in the traditonal sense) and thus only the $U(\mathfrak{g})$-module structure on V^∞ is not sufficient to reconstruct the G-module structure on V. This is mainly a topological obstruction. Notice however, that we have a small work-around for this by using the Polar (Cartan) decomposition of G. It shows that $K \simeq G$ and thus morally the representation theory of G should come from the representation theory of K, possibly with some twist. As it will turn out, this yoga is true for irreducible representations! As understanding the irreducible representations of G are the real goal of representation theory, this is exactly the result we want.

The "twist" mentioned above is that we shall consider a certain subspace of V which is generated by suitably nice vectors which allow us to decompose V as an algebraic direct sum of irreducible spaces. This will in turn be the correct space to study.

Definition 2.2.17. A vector $v \in V$ is called K-**finite** if $\Pi(K)v$ spans a finite dimensional vector space. The subspace of all K-finite vectors is denoted V_K.

If (Π, V) is a representation of G, then we can investigate $\Pi|_K$. If K acts by unitary operators, then $(\Pi|_K, V)$ decomposes as a Hilbert space direct sum of irreducible representations of K. For each $\gamma \in \widehat{K}$, denote by $V(\gamma)$ the sum of all of the subrepresentations isomorphic to γ. This is called the $\gamma -$ **isotypic component** of V.

Definition 2.2.18. A representation of G is called **admissible** if $\dim_{\mathbb{C}} V(\gamma) < \infty$ for all $\gamma \in \widehat{K}$. Notice that this is equivalent to $m_\gamma(\Pi|_K) < \infty$.

Proposition 2.2.19. *Let (Π, V) be an admissible representation of G. Then $V_K \subseteq V^\infty$.*

Proof. We shall proceed in a few steps.

Step 1: Notice that $V_K \cap V^\infty \neq \varnothing$ as V^∞ is dense in V. We shall momentarily call vectors in this intersection: smooth K-finite vectors for V. Let us show that a special class of functions give K-finite vectors. Consider λ the left-regular representation of K on $C^\infty(K)$. Denote by λ_{fin} the K-finite vectors in $C^\infty(K)$ and let $f \in \lambda_{\text{fin}}$. Further, let $h \in C_c^\infty(\exp \mathfrak{p}_0)$ and put

$$F(k \exp \mathfrak{p}_0) = f(k)h(\exp X)$$

Each F of this form is compactly supported and *left* K-finite for λ as

$$\lambda(k_0)F(k \exp X) = F(k_0^{-1}k \exp X) = f(k_0^{-1}k)h(\exp X)$$

where the right hand side is K-finite by assumption.

Now, for every $v \in V$, $\Pi(F)v \in V^\infty$ by the proof of Proposition 2.2.16. Unraveling the definitions a bit, we see that

$$\Pi(k_0)\Pi(F)v = \int_G F(g)\Pi(k_0 g)v\, dg = \int_G F(k_0^{-1}g)\Pi(g)v\, dg = \lambda(k_0^{-1})\Pi(F)v$$

with the right hand side K-finite by above. Hence, $\Pi(F)v$ is K-finite.

Step 2: Now we shall show that the linear span of all $\Pi(F)v$ is dense in V. Let $v \in V$ be arbitrary. As Π is continuous, the set

$$T = \{g \in G : ||\Pi(g)v - v|| < \varepsilon\}$$

is open and contains some compact subset C. We may then choose f, h supported in C such that $\int_G F(g)\, dg = 1$. Then

$$\begin{aligned} ||\Pi(F)v - v|| &= \left|\left| \int_G F(g)\Pi(g)v\, dg - v \right|\right| \\ &\leq \int_G F(g)\, ||\Pi(g)v - v||\, dg \\ &< \varepsilon \int_G F(g)\, dg = \varepsilon \end{aligned}$$

Hence, the linear span of all such $\Pi(F)v$ is dense in V. This shows that the set of smooth K-finite vectors (namely $V_k \cap V^\infty$) is dense in V.

Step 3: Now we show that for admissible representations this subspace this is precisely all of the K-finite vectors. As V is assumed admissible, write $\Pi_K = \widehat{\bigoplus}_{\gamma \in \widehat{K}} V(\gamma)$ with each $\dim_{\mathbb{C}} V(\gamma) < \infty$. Consider $V(\gamma)$. As this is finite dimensional and irreducible, it consists entirely of K-finite vectors. As this is true for every such γ, we see that $V_K = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$ (the algebraic direct sum). Combining this with the result from above, we have a dense subspace of a linear space which in turn must be the entire space. Since, the K-finite vectors are finite linear combinations of elements of the $V(\gamma)$, we see that $V_K = V_K \cap V^\infty$. Hence, all K-finite vectors are smooth. □

Similar to the smooth case above. Harish-Chandra proved that all K-finite vectors are in fact real analytic by way of matrix coefficients and elliptic differential operators. The main tool in that approach is the following fact which we shall not prove:

Theorem 2.2.20. *Let D be an elliptic differential operator with real-analytic coefficients. Then every element in the solution space $Df = 0$ is real-analytic.*

It can be shown that K-finite matrix coefficients satisfy a certain elliptic differential operator and thus are real-analytic. For a proof of this, see [Kna86]

Lemma 2.2.21. *V_K is a $\mathfrak{g}$ invariant subspace of V^∞.*

Proof. Let $v \in V_K$ and put $W_v = \pi(U(\mathfrak{k}_{\mathbb{C}}))v$ the necessarily finite dimensional subspace corresponding to v. Then for $X \in \mathfrak{k}, Y \in \mathfrak{g}$, and $v' \in W_v$, we have that

$$\pi(X)\pi(Y)v' = \pi(Y)\pi(X)v' - \pi([X, Y])v'$$

and $\pi(\mathfrak{g})v'$ is stable under $\pi(\mathfrak{k})$. As W_v is finite dimensional, we can exponentiate the elements and conclude that $\pi(\mathfrak{g})W_v$ is $\Pi(K)$-invariant. Thus, $Y \in \mathfrak{g}$ implies that $\pi(Y)v$ is an element of a finite dimensional vector space which is $\Pi(K)$-stable. Hence, $\pi(Y)v$ is K-finite □

Therefore V_K carries two related representations: one of $\mathfrak{g}$ and one of K. As it turns out, vector spaces with this property are incredibly rich in structure and thus are the next object of study. We formalize this in the following definition.

Definition 2.2.22. (Lepowsky) Let G be a non-compact reductive Lie group, $G = KAN$ the Iwasawa decomposition. A $(\mathfrak{g}, K)$-**module** is a vector space V equipped with two representations, denoted by $\pi : \mathfrak{g} \cup K \to \mathrm{End}(V)$, of $\mathfrak{g}$ and K, such that the following conditions are satisfied:

(a) Every $v \in V$ is K-finite.

(b) The differential of $\pi|_K$ is the restriction of $\pi_{\mathfrak{g}}|_{\mathfrak{k}_0}$.

(c) For all $k \in K$ and $X \in \mathfrak{g}$, we have that $\pi(\mathrm{Ad}(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$.

If V is equipped with an inner product, we say that V is a **unitary** $(\mathfrak{g}, K)$-module if $\pi(X)$ is a unitary operator for all $X \in \mathfrak{g}$. We say that V is an **admissible** $(\mathfrak{g}, K)$-module if for every $\gamma \in \widehat{K}$, the γ-isotypic component $V(\gamma)$ is finite dimensional.

Corollary 2.2.23. *If (Π, G) is an admissible representation of G, then (π, V_K) is a $(\mathfrak{g}, K)$-module by Lemma 2.2.21.*

2.3 $(\mathfrak{g}, K)$-modules

This next section will cover the basics of the $(\mathfrak{g}, K)$-module theory. In particular, we shall see that irreducible $(\mathfrak{g}, K)$-modules completely determine the irreducible G-modules. This is the correspondence hinted at in the introduction. We have not proven this directly, but it can be shown that the assignment $V \mapsto V_K$ is functorial and this functor is fully faithful on the category of irreducible $(\mathfrak{g}, K)$-modules (up to equivalence).

2.3.1 Admissible and Unitary representations

We have mirrored the finite dimensional theory fairly closely to this junction. One result which has been starkly absent is Schur's Lemma. The main reason for this is that in the infinite dimensional setting, it may not hold! Luckily, if we restrict ourselves to *countably* infinite bases, we get an analogous result due to Dixmier.

Lemma 2.3.1. *Let V be a countable dimensional $\mathbb{C}$-vector space and suppose $T \in \mathrm{End}(V)$. Then there exists a scalar $c \in \mathbb{C}$ such that $T - cI$ is not invertible.*

Proof. Suppose for the sake of contradiction that $T - cI$ is invertible for all $c \in \mathbb{C}$. Then for every polynomial P, $P(T)$ is invertible as a function of one variable. Now, let $R(T) = P(T)/Q(T)$ be some rational function. This gives a linear map $\mathbb{C}(x) \to \mathrm{End}(V)$. For all $v \in V$, we have that $R(T)v = 0$ only if $P(T)v = 0$. Thus the map $\mathbb{C}(x) \to V$ given by $R \mapsto R(T)v$ is injective. Since $\mathbb{C}(x)$ is uncountably infinite dimensional over $\mathbb{C}$, we have a contradiction. $\square$

Lemma 2.3.2. *(Dixmier) Suppose $S \subsetneq \mathrm{End}(V)$ is a subset of endomorphisms which acts irreducibly. If $T \in \mathrm{End}(V)$ commutes with all elements of S, then $T = cI$ for some $c \in \mathbb{C}$.*

Proof. By the preceding lemma, there exists some $c \in \mathbb{C}$ such that $T - cI$ is not invertible. Consider $\ker(T - cI)$ and $\mathrm{Im}(T - cI)$. Every element of S preserves both of these spaces one of them is necessarily a proper subset. As $T - cI$ is not invertible, $\ker(T - cI)$ is nonzero. Thus $\ker(T - cI) = V$ as all of the elements of S act irreducibly. Hence $T - cI = 0$ and $T = cI$. $\square$

Definition 2.3.3. If V and W are two $(\mathfrak{g}, K)$-modules, denote by $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ the set of all $\mathfrak{g}$ homomorphisms $V \to W$ which are also K homomorphisms.

Lemma 2.3.4. *Let V be an irreducible $(\mathfrak{g}, K)$-module. Then $\mathrm{Hom}_{\mathfrak{g},K}(V, V) = \mathbb{C} \cdot \mathrm{Id}$.*

Proof. Let $v \in V$ and W_v the span of v under K. Then $U(\mathfrak{g})W_v$ is a $\mathfrak{g}$-invariant, K-invariant subspace and thus $V = U(\mathfrak{g})W_v$. This exhibits V as a countably infinite dimensional space. By applying the above lemmas, we are done. □

The main motivation for studying representations began with Harmonic analysis where unitary representations arise naturally. In fact, all of the theory developed so far was originally done precisely to understand the *Unitary dual* of reductive Lie groups. Based on the discussion above, one may guess that admissible representations are related to unitary representations in some way. In fact, the precise statement is as follows:

Theorem 2.3.5. *Let G be a real reductive Lie group and (π, V) an irreducible unitary representation of G. Then (π, V) is admissible as a representation of G and thus V_K is an admissible, unitary $(\mathfrak{g}, K)$-module.*

Before getting to the proof, we need a few technical results, some of which we shall not prove for brevity. All of the statements can be found in [HC53]. Let $U(\mathfrak{k})$ denote the enveloping algebra of $(\mathfrak{k}_0)_{\mathbb{C}}$ considered as a subalgebra of $U(\mathfrak{g})$ and $\mathfrak{Y}$ a left ideal in $U(\mathfrak{k})$ such that $U(\mathfrak{k})/\mathfrak{Y}$ is finite dimensional and $\mathfrak{k}$ acts by semisimple transformations. Consider $X = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{Y}$. $U(\mathfrak{g})$ acts on this space by left-translation and thus restricts to a representation of $Z(\mathfrak{g})$ (the center of $U(\mathfrak{g})$. Additionally, using the adjoint representation, we get an action of K on this quotient. Let $X(\gamma)$ denote the γ-isotypic component of X.

Lemma 2.3.6. *For every $\gamma \in \widehat{K}$, $X(\gamma)$ is a finite module over $Z(\mathfrak{g})$.*

Proof. See [HC53, Theorem 1]. □

Lemma 2.3.7. *Let (π, V) be an admissible representation of G and V_K the associated $(\mathfrak{g}, K)$-module with $\psi \in V_K$. Then the closure $\overline{\pi(U(\mathfrak{g})\psi)}$ is $\pi(G)$ invariant.*

Proof. Let $\psi_0 \in U = \pi(U(\mathfrak{g})\psi$, and $\lambda \in V^*$ such that λ vanishes on U. Since $\psi_0 \in V_K$, we see that the map

$$g \mapsto \lambda(\pi(g)\psi_0)$$

is real analytic on G. Now, there exists a neighbourhood O of 0 in $\mathfrak{g}_0$ such that the exponential map is given by a power series and thus

$$\pi(\exp X)\psi_0 = \sum_{m=0}^{\infty} \frac{1}{m!}\pi(X)^m\psi_0$$

and thus

$$\lambda(\pi(\exp X))\psi_0) = \sum_{m=0}^{\infty} \frac{1}{m!}\lambda(\pi(X)^m\psi_0)$$

As $\pi(X)^m\psi_0 \in U$, the right hand side is 0. Therefore, $\lambda(\pi(g)\psi_0)$ vanishes on a neighbourhood of 1 in G and by analyticity, vanishes everywhere. Applying the Hahn-Banach theorem, we see that $\pi(g)\psi_0 \in \overline{U}$. Therefore $\pi(g)U \subseteq \overline{U}$ and by continuity, $\pi(g)\overline{U} \subseteq \overline{U}$. This completes the proof. □

Lemma 2.3.8. *Let (π, V) be a representation of G which admits an infinitesimal character. Then for any K-invariant subspace W, put $W(\gamma) = W \cap V(\gamma)$. If $\bigoplus_{\gamma \in \widehat{K}} W(\gamma)$ is dense in V, then $V(\gamma) = \overline{W(\gamma)}$.*

Proof. See [HC53, Lemma 30]. □

Remark 2.3.9. For a moral proof of the above lemma, mimic the ideas of the proof that the Gårding subspace is dense. Consider the integral operators associated to certain smooth functions and use the density of these to conclude the desired lemma.

Proposition 2.3.10. *Let (π, V) be a representation of G on a Hilbert space which admits an infinitesimal character. Let $\psi_0 \in V_K = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$ and $U = \overline{\pi(U(\mathfrak{g})\psi_0)}$. Then U is invariant under $\pi(G)$, $\pi(U(\mathfrak{g}))\psi_0 = \bigoplus_{\gamma \in \widehat{K}} U(\gamma)$, and $\dim_{\mathbb{C}} U(\gamma) < \infty$.*

Proof. We know from Lemma 2.3.7 that U is $\pi(G)$ invariant. Put $U_0 = \pi(U(\mathfrak{g}))\psi_0$. Then $U_0 \subseteq V_K$ and therefore $U_0 = \bigoplus_{\gamma \in \widehat{K}} U_0 \cap V(\gamma)$. Let $\mathfrak{Y}$ be the set of all elements $x \in U(\mathfrak{k})$ such that $\pi(x)\psi_0 = 0$. Then $\mathfrak{Y}$ is a left ideal in $U(\mathfrak{k})$ and it satisfies the conditions prior to Lemma 2.3.6. Set $X = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{Y}$. Put $(\tilde{\pi}, X)$ the associated representation of $U(\mathfrak{g})$ on X. Define the map

$$\alpha : X \to U_0 \qquad \alpha([x]) = \pi(x)\psi_0$$

where x is a representative of $[x]$. This is well defined as for any two representative $x, x' \in [x]$, we have that $x = x' + y$ where $y \in \mathfrak{Y}$. Now $\alpha(x) = \alpha(x') + \alpha(y) = \alpha(x')$ by the definition of $\mathfrak{Y}$. Now, for every $a \in U(\mathfrak{g})$, we have that $\alpha(\tilde{\pi}(a)[b]) = \pi(a)\alpha([b])$. Further, α is injective and thus by identifying X with its image in U_0, can consider $X \subseteq U_0$ and thus we have a direct sum decomposition

$$X = \bigoplus_{\gamma \in \widehat{K}} X(\gamma)$$

and $\alpha(X(\gamma)) = U_0 \cap V(\gamma)$. Applying Lemma 2.3.6, we get that each $X(\gamma)$ is a finite module over $Z(\mathfrak{g})$. Pick $[b_1], [b_2], ..., [b_k]$ elements in $X(\gamma)$ such that $X(\gamma) = \bigoplus_i \tilde{\pi}(Z(\mathfrak{g}))[b_i]$. Hence, $U_0 \cap V(\gamma) = \bigoplus_i \pi(Z(\mathfrak{g}))\alpha([b_i])$. Now for each element of $z \in Z(\mathfrak{g})$,

$$\pi(z) = \chi(z)\pi(1)$$

and thus the $\alpha([b_i])$ span $U_0 \cap V(\gamma)$ and thus $\dim U_0 \cap V(\gamma) < \infty$. Since $U_0 = \bigoplus_{\gamma \in \widehat{K}} U_0 \cap U(\gamma)$ is dense in U, by Lemma 2.3.8 we conclude that $U(\gamma) = U_0 \cap U(\gamma)$ and each of these is finite dimensional. This concludes the proof. □

Proof of Theorem 2.3.5. It is known that every irreducible unitary representation has an infinitesimal character. As the K-finite vectors are analytic and dense in V. Pick $\psi_0 \in V_K$ non-zero. Then by irreducibility, $V = \overline{\pi(U(\mathfrak{g}))\psi_0}$ and applying the above proposition, we conclude that $\dim V_K(\gamma) < \infty$. Hence, every irreducible unitary representation is admissible. □

The remaining part of this section will encompass the proof that irreducibility of admissible representations can be checked on either the group or $(\mathfrak{g}, K)$ level. We will prove this by way of considering the K-finite matrix coefficients for the given representations. In particular, the main tool is actually a fact from the theory of partial differential equations which we quote below:

Theorem 2.3.11 (Regularity Theorem)**.** *Let D be an elliptic differential operator on $C^\infty(G)$. Then if the coefficients of D are real analytic and u is a solution to $Du = 0$, then u is real analytic.*

Proof. [Gru09, Theorem 6.29]. $\square$

What we shall show is that every K-finite matrix coefficient is annihilated by an elliptic differential operator and thus by the previous theorem, every K-finite matrix coefficient is real analytic.

Proposition 2.3.12. *Let G be a real reductive group and (π, V) an admissible representation of G. Then every matrix coefficient of the form $g \mapsto (\pi(g)u, v)$ for $u \in V_K$ is real analytic.*

Proof. By unraveling the definitions, we see that for any $D \in U(\mathfrak{g})$, we have that

$$D(\pi(g)u, v) = (\pi(g)\pi(D)u, v)$$

We may assume without loss of generality that u is contained in some $V(\gamma)$. As π is admissible this is finite dimensional and there exists $c_1, ..., c_n \in \mathbb{C}$ such that

$$\prod_i \pi(\Omega) - c_i = 0$$

where Ω is the Casimir element in $Z(\mathfrak{g})$. If we denote by Ω_K the Casimir element of $Z(\mathfrak{k}_\mathbb{C})$ then $\pi(\Omega_K) = c_\gamma$ on $V(\gamma)$ by Schur's Lemma. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan Decomposition of the Lie algebra and pick bases for $\mathfrak{k}_0$ and $\mathfrak{p}_0$ which are orthogonal with respect to the inner product $B_\theta(X, Y) = -B(X, \theta Y)$ where θ is the Cartan involution. Then we have the following equalities

$$\begin{aligned}
\Omega &= -\sum X_i^2 + \sum Y_i^2 \\
\Omega_K &= -\sum X_i^2 \\
\Omega - 2\Omega_K &= \sum X_i^2 + \sum Y_i^2
\end{aligned}$$

This is seen to be an elliptic differential operator by investigating its principal symbol in a neighbourhood of the identity using coordinates defined by the exponential function. From this, we see that the differential operator

$$D = \prod_i \Omega - 2\Omega_K + 2c_\gamma - c_j$$

is also elliptic with real analytic coefficients (as K is an analytic manifold). Now, from our equation above,

$$\begin{aligned}
D(\pi(g)u, v) &= \left(\pi(g) \prod_i \left[\pi(\Omega) - \pi(2\Omega_K) + 2c_\gamma - c_j\right] u, v\right) \\
&= \left(\pi(g) \prod_i \left[\pi(\Omega) - c_j\right] u, v\right) \\
&= 0
\end{aligned}$$

Hence, by the Regularity Theorem $(\pi(g)u, v)$ is real analytic. $\square$

Definition 2.3.13. Let (π, V) be a representation of G on a HIlbert (Banach,Fréchet, etc.) space and V_K the associated $(\mathfrak{g}, K)$-module. Then the **contragredient representation** or **dual representation** of V_K is denoted as $V_K^{\star}$ and is defined to be

$$V_K^{\star} = (V_K^{*})_K$$

There is a natural transpose action on $V_K^{\star}$ and from this we see that all linear functionals $(-, v)$ for v K-finite are contained in V_K^{*}. In fact, this is the entire space!

Corollary 2.3.14. *Let (π, V) and (π', V') be irreducible admissible representations of G. If π and π' are infinitesimally equivalent (V_K and V_K' are isomorphic $(\mathfrak{g}, K)$-modules) then they have the same set of matrix coefficients.*

Proof. The matrix coefficients on G are characterized as the unique real analytic functions such that their derivative at $g = 1$ is given by

$$D(\pi(g)u, v) = (\pi(D)u, v)$$

From the discussion in the definition of the contragredient representation, we see that this is real analytic and given by $(\pi(D)u, v')$ for $v' \in V_K^{\star}$ not depending on D. Therefore, the matrix coefficients are given in a way which is infinitesimally independent. This completes the proof. □

Corollary 2.3.15. *The closed G-invariant subspaces of V are in one-to-one correspondence with the $\mathfrak{g}$-invariant subspaces of V_K with the correspondence given by*

$$U \mapsto U \cap V_K$$

and

$$\overline{W} \leftarrow W$$

Proof. By Lemma 2.3.7, we have that $U \cap V_K$ is $\mathfrak{g}$-invariant and $\overline{W}$ is $\pi(G)$-invariant. Thus, it remains to prove that these operations are inverses. For any closed invariant subspace U, put $U_K = U \cap V_K$. Then $\overline{U_K} \subseteq U$ as U is closed. Furthermore, U_K is dense in U and thus $\overline{U_K} = U$. This completes the proof. □

Theorem 2.3.16. *Let (π, V) be an admissible representation of the reductive Lie group G. Then V is irreducible if and only if V_K is an irreducible $(\mathfrak{g}, K)$-module.*

Proof. This is a particular case of the previous corollary. □

Corollary 2.3.17. *Let (π, V) and (π', V') be irreducible admissible representations of G. If V and V' share a single matrix coefficient in common, then $V_K \cong_{\mathfrak{g},K} V_K'$.*

Proof. Our assumption is that

$$(\pi(g)u, v) = (\pi'(g)u', v')$$

for all $g \in G$ and some non-zero $u, v \in V_K$ and $u', v' \in V_K'$. Let $V_0 = U(\mathfrak{g})(\pi(g)u, v)$ be a subspace of $C^{\infty}(G)$. By the previous theorem, we have that $\pi(U(\mathfrak{g}))u = V_K$. and thus

$$V_0 = (\pi(g)\pi(U(\mathfrak{g}))u, v) = (\pi(g)V_K, v)$$

Define $\varphi : V_K \to V_0$ by $v \mapsto (\pi(-)\pi(U(\mathfrak{g}))u, v)$. This map is onto by construction. Further, it is $U(\mathfrak{g})$-equivariant since

$$\begin{aligned}\varphi(\pi(D)(\pi(D')u)) = \varphi(\pi(DD')u) &= (\pi(-)\pi(DD')u, v) \\ &= D(\pi(-)\pi(D')u, v) \\ &= D\varphi(\pi(D')u)\end{aligned}$$

Since π is irreducible, $\ker \varphi = 0$ and thus it is a $U(\mathfrak{g})$-module isomorphism.

Starting with π' instead, we get a $U(\mathfrak{g})$-module isomorphism $\psi : V'_K \to V_0$ and thus by taking $\psi^{-1}\varphi$ be have the desired $(\mathfrak{g}, K)$-module isomorphism. □

Corollary 2.3.18. *Consider the induced representations* $V = L^2(G, V_\sigma, \sigma), W = C^\infty(G, V_\sigma, \sigma), U = C(G, V_\sigma, \sigma)$ *where the latter two are defined naturally. Then* $V_K \cong W_K \cong U_K$.

Proof. Modulo the result that these representations are admissible, we see that for any smooth, L^2 function $G \to V$ we have that the matrix coefficient corresponding to this element will be equal in all of the above representations. Now by the previous corollary, we conclude the result. □

This remedies our concern from above that taking different types of induction would give substantive differences in the representation theory. Precisely all of the analytic concerns are then removed from the study of the underlying $(\mathfrak{g}, K)$-modules and this is where we begin the final steps towards the Langlands classification.

Bibliography

[ABVJ92] Jeffrey Adams, Dan Barbasch, and David A. Vogan Jr. *The Langlands Classification and Orreducible Characters for Real Reductive Groups*, volume 104 of *Progress in Mathematics*. Birkhäuser Boston, 1992.

[Gru09] Gerd Grubb. *Distributions and Operators*, volume 252 of *Graduate Texts in Mathematics*. Springer, 2009.

[Hal15] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations*, volume 222 of *Graduate Texts in Mathematics*. Springer International Publishing. Switzerland 2013, 2015.

[HC53] Harish-Chandra. Representations of semisimple lie groups on a banach space i. *Trans. American Mathematical Society*, 75:185–243, 1953.

[Hel78] Sigurdur Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, 1978.

[Kna86] Anthony W. Knapp. *Representation Theory of Semisimple Groups: An Overview Based on Examples (PMS-36)*. Princeton University Press, rev - revised edition, 1986.

[Kna05] Anthony K. Knapp. *Lie Groups Beyond an Introduction*, volume 140 of *Progress in Mathematics*. Springer Sceince+Business Media, 2nd edition, 2005.

[KVJ95] Anthony W. Knapp and David A. Vogan Jr. *Cohomological Induction and Unitary Representations*. Princeton University Press, 1995.

[Lan89] Robert P. Langlands. On the classification of irreducible representations of real algebraic groups. In *Representation Theory and Harmonic Analysis on semisimple Lie groups*, volume 31 of *Mathematical Surveys and Monographs*, pages 101–170. American Mathematical Society, 1989.

[VJ81] David A. Vogan Jr. *Representations of Real Reductive Groups*, volume 15 of *Progress in Mathematics*. Birkhäuser Boston, 1981.

[Wal88] Nolan R. Wallach. *Real Reductive Groups I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, 1988.