MATH 2270

Exam #1 - Fall 2008

Name: Answer key

1. Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 3 & 7 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$.

   (a) Solve the linear system $A\vec{x} = \vec{b}$ for $\vec{x}$ using Gauss-Jordan elimination. Clearly show each step and indicate your answer. If the system is inconsistent, write *inconsistent*.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 7 & 5 \end{array}\right) \begin{array}{c} \\ -\mathrm{I} \\ -\mathrm{I} \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & 1 & 3 & 2 \end{array}\right) \begin{array}{c} \\ \times -1 \\ \\ \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{array}\right) \begin{array}{c} -2\mathrm{II} \\ \\ -\mathrm{II} \end{array} \longrightarrow$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

x_3 is a free variable. Set $x_3 = t$.

$$x_1 - 2x_3 = -1$$
$$x_2 + 3x_3 = 2$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t-1 \\ -3t+2 \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

$\therefore$ the set of all vectors of the form

$$\boxed{t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}.$$

(b) what is rref(A) (i.e. the reduced row echelon form of A)?

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}(A)$$

(c) what is rank(A)? Why?

rank(A) = 2 b/c there are two leading 1's in rref(A).

(d) Is the matrix A invertible? Why?

No, b/c rref(A) $\neq I_3$.

(e) Find a basis for the kernel of A (i.e. ker(A)).

$x_1 - 2x_3 = 0$

$x_2 + 3x_3 = 0$

let $x_3 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +2t \\ -3t \\ t \end{pmatrix} = \boxed{t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}}$$

$\therefore$ the vector $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ is a basis for the 1-dim. kernel of A.

(f) Find a basis for the image of A (i.e. im(A)).

A basis for the image of A is given by the columns of A corresponding to leading 1s in rref(A). In this case we have: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Need to check T ~~the~~ ~~the~~ distributes over addition and scalar multiplication.

(a) Prove T is a linear transformation.

- $T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ ✓

- $T\left(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) = \begin{pmatrix} \alpha x_1 \\ \alpha y_1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} = \alpha T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ ✓

(b) Write the matrix corresponding to T.

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\therefore$ we have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(c) Give an example of a vector in $\mathbb{R}^3$ not contained in either $\operatorname{im}(T)$ or $\ker(T)$.

$\operatorname{im}(T) = x\text{-}y$ plane

$\ker(T) = z$-axis

$\therefore$ we can take $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

3. Let $k \in \mathbb{R}$. Compute the inverse of the following matrix using Gauss-Jordan elimination. Show your work.

$$A = \begin{pmatrix} 1 & k \\ 0 & 2 \end{pmatrix}.$$

$$\left(\begin{array}{cc|cc} 1 & k & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \% \tfrac{1}{2} \end{array} \longrightarrow \left(\begin{array}{cc|cc} 1 & k & 1 & 0 \\ 0 & 1 & 0 & \tfrac{1}{2} \end{array}\right) \begin{array}{l} -k(\mathrm{II}) \\ \\ \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -k/2 \\ 0 & 1 & 0 & 1/2 \end{array}\right). \quad \therefore \quad A^{-1} = \begin{pmatrix} 1 & -k/2 \\ 0 & 1/2 \end{pmatrix}$$

check:

$$\begin{pmatrix} 1 & k \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -k/2 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

4. Let $a, b, c, d \in \mathbb{R}$. Compute the following matrix products.

$2\times 1 \quad 1\times 2$

(a) $\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ca & da \\ cb & db \end{pmatrix}$

$1\times 2 \quad 2\times 1$

(b) $\begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca + db \end{pmatrix}$

5. Consider the $n \times 4$ matrix

$$A = \begin{pmatrix} | & | & | & | \\ \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} \\ | & | & | & | \end{pmatrix}.$$

You are told the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ is an element of $\ker(A)$. Write $\vec{v_4}$ as a linear combination of the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

Since $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \in \ker(A)$, we have

$$\vec{v_1} + 2\vec{v_2} + 3\vec{v_3} + 4\vec{v_4} = \vec{0}$$

$$\therefore \quad \vec{v_4} = -\frac{1}{4}\left(\vec{v_1} + 2\vec{v_2} + 3\vec{v_3}\right)$$

$$= -\frac{\vec{v_1}}{4} - \frac{\vec{v_2}}{2} - \frac{3\vec{v_3}}{4}$$

6. Multiple choice — choose the best answer for the following questions.

(a) Suppose two distinct solutions, $\vec{x_1}$ and $\vec{x_2}$, can be found for the linear system $A\vec{x} = \vec{b}$. Which of the following is necessarily true?

i. $\vec{b} = \vec{0}$.

ii. A is invertible.

iii. A has more columns that rows.

iv. $\vec{x_1} = -\vec{x_2}$

v. There exists a solution $\vec{x}$ such that $\vec{x} \neq \vec{x_1}$ and $\vec{x} \neq \vec{x_2}$.

(b) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation whose kernel is a 3-dimensional subspace of $\mathbb{R}^5$. Then $\text{im}(T)$ is

i. The trivial subspace.

ii. A line through the origin.

iii. A plane through the origin.

iv. all of $\mathbb{R}^3$.

v. Cannot be determined from the given information.

7. Compute the following matrix product.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{100} - \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{100}.$$

Note: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$

So $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{100} = \begin{pmatrix} 1 & 200 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{100} = \begin{pmatrix} 1 & 0 \\ 200 & 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{100} - \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{100} = \begin{pmatrix} 0 & 200 \\ -200 & 0 \end{pmatrix}$