MATH 2270

Exam #2 - Fall 2008

Name: ______________________________

1. (8 points) Let

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix}; \ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.

(a) If $\vec{x} = \vec{v}_1 - \vec{v}_2 + \vec{v}_3$, find $(\vec{x})_{\mathfrak{B}}$.

(b) Observe $A\vec{v}_1 = 9\vec{v}_1$. Find $(A\vec{v}_1)_{\mathfrak{B}}$.

(c) Find $(A\vec{v}_3)_{\mathfrak{B}}$.

(d) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B}$. (HINT: Proceed column by column — don't try to take an inverse).

2. (8 points) Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation given by

$$T(M) = AM$$

where A is a fixed *invertible* matrix in $\mathbb{R}^{2\times 2}$.

(a) Find the image of T.

(b) Find the rank of T.

(c) Find the kernel of T.

(d) Find the nullity of T.

3. (6 points) Let V be the linear space spanned by the functions $\mathfrak{B} = \{\cos(t), \sin(t)\}$ and let $T : V \to V$ be the linear transformation given by

$$T(f) = f'' + 2f' + 3f.$$

Find the matrix of T with respect to the basis $\mathfrak{B}$. (HINT: recall $\sin(t)' = \cos(t)$ and $\cos(t)' = -\sin(t)$).

4. (8 points) Find the QR factorization of the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Clearly show each step.

5. (6 points) Recall the *trace* of a square matrix is the sum of its diagonal entries. For example,

$$\operatorname{trace}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 + 4 = 5.$$

Let $V = \mathbb{R}^{2\times 2}$ be the inner product space with inner product

$$\langle A, B\rangle = \operatorname{trace}\left(A^T B\right).$$

(a) If $A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$, find $\langle A, B\rangle$.

(b) Is A orthogonal to B in V?

(c) Normalize A (i.e. find an element of unit length in the direction of A in V).

(d) What is the norm of an orthogonal matrix in V?

6. (4 points) Multiple choice. Choose the best answer to each of the following questions.

(a) If $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{2\times 2}$ is a linear transformation, then the kernel of T must be at least

i. 1-dimensional.

ii. 2-dimensional.

iii. 3-dimensional.

iv. 4-dimensional.

v. 5-dimensional.

(b) If $\vec{v}$ is a unit vector in $\mathbb{R}^n$, then the $n \times n$ matrix $\vec{v}\vec{v}^T$ has rank

i. 0.

ii. 1.

iii. n.

iv. $n-1$.

v. cannot be determined from the given information.