**MATH 2270**
Exam #2 - Fall 2008

Name: Answer Key

1. (8 points) Let

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix}; \ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$.

(a) If $\vec{x} = \vec{v}_1 - \vec{v}_2 + \vec{v}_3$, find $(\vec{x})_{\mathfrak{B}}$.

$$(\vec{x})_{\mathfrak{B}} = \boxed{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}_{\mathfrak{B}}}$$

(b) Observe $A\vec{v}_1 = 9\vec{v}_1$. Find $(A\vec{v}_1)_{\mathfrak{B}}$.

$$(A\vec{v}_1)_{\mathfrak{B}} = \boxed{\begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}_{\mathfrak{B}}}$$

(c) Find $(A\vec{v}_3)_{\mathfrak{B}}$.

$$\begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \boxed{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{\mathfrak{B}}}$$

(d) Find the matrix $B$ of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B}$. (HINT: Proceed column by column — don't try to take an inverse).

$$(A\vec{v}_2) = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{\mathfrak{B}}.$$

$$\therefore \quad \mathfrak{B} = \begin{pmatrix} | & | & | \\ T(\vec{v}_1)_{\mathfrak{B}} & T(\vec{v}_2)_{\mathfrak{B}} & T(\vec{v}_3)_{\mathfrak{B}} \\ | & | & | \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 9 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

2. (8 points) Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the linear transformation given by

$$T(M) = AM$$

where $A$ is a fixed *invertible* matrix in $\mathbb{R}^{2\times 2}$.

(a) Find the image of $T$.

If $B \in \mathbb{R}^{2\times 2}$, then $AM = B \iff M = A^{-1}B$. Therefore $T$ is surjective and $\operatorname{im}(T) = \mathbb{R}^{2\times 2}$.

(b) Find the rank of $T$.

$\operatorname{rank}(T) = \dim(\operatorname{im}(T)) = \dim(\mathbb{R}^{2\times 2}) = 2\cdot 2 = \boxed{4}$.

(c) Find the kernel of $T$.

$AM = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff M = A^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore

$\ker(T) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$.

(d) Find the nullity of $T$.

$= \dim(\ker(T)) = \boxed{0}$.

3. (6 points) Let $V$ be the linear space spanned by the functions $\mathfrak{B} = \{\cos(t), \sin(t)\}$ and let $T : V \to V$ be the linear transformation given by

$$T(f) = f'' + 2f' + 3f.$$

Find the matrix of $T$ with respect to the basis $\mathfrak{B}$. (HINT: recall $\sin(t)' = \cos(t)$ and $\cos(t)' = -\sin(t)$).

$$B = \begin{pmatrix} | & | \\ T(\cos(t))_{\mathfrak{B}} & T(\sin(t))_{\mathfrak{B}} \\ | & | \end{pmatrix}$$

$$\begin{aligned} T(\cos(t)) &= \cos''(t) + 2\cos'(t) + 3\cos(t) \\ &= -\cos(t) - 2\sin(t) + 3\cos(t) \\ &= 2\cos(t) - 2\sin(t) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}_{\mathfrak{B}}. \end{aligned}$$

$$\begin{aligned} T(\sin(t)) &= \sin''(t) + 2\sin'(t) + 3\sin(t) \\ &= -\sin(t) + 2\cos(t) + 3\sin(t) \\ &= 2\sin(t) + 2\cos(t) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{\mathfrak{B}}. \end{aligned}$$

$$\therefore \boxed{B = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}}$$

4. (8 points) Find the $QR$ factorization of the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Clearly show each step.

Let $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\vec{v}_1}{1} = \vec{v}_1$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \underbrace{(\vec{v}_2 \cdot \vec{u}_1)}_{=1} \cdot \vec{u}_1 = \vec{v}_2 - \vec{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \vec{u}_2}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \underbrace{(\vec{v}_3 \cdot \vec{u}_1)}_{=1} \cdot \vec{u}_1 - \underbrace{(\vec{v}_3 \cdot \vec{u}_2)}_{=1} \cdot \vec{u}_2$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{u}_3}$$

$\therefore$

$$M = QR = \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

5. (6 points) Recall the *trace* of a square matrix is the sum of its diagonal entries. For example,

$$\operatorname{trace}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 + 4 = 5.$$

Let $V = \mathbb{R}^{2\times 2}$ be the inner product space with inner product

$$\langle A, B\rangle = \operatorname{trace}\left(A^T B\right).$$

(a) If $A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$, find $\langle A, B\rangle$.

$$\langle A, B\rangle = \text{trace}\left(\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}\right) = \text{trace}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0 + 0 = \boxed{0}$$

(b) Is $A$ orthogonal to $B$ in $V$?

Yes, b/c $\langle A, B\rangle = 0$.

(c) Normalize $A$ (i.e. find an element of unit length in the direction of $A$ in $V$).

$$\|A\|^2 = \langle A, A\rangle = \text{trace}\left(\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\right) = \text{trace}\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = 4 + 4 = \boxed{8}$$

$$\therefore \|A\| = \sqrt{8} \text{ and } \frac{A}{\|A\|} = \frac{A}{\sqrt{8}} = \boxed{\begin{pmatrix} 2/\sqrt{8} & 2/\sqrt{8} \\ 0 & 0 \end{pmatrix}}$$

(d) What is the norm of an orthogonal matrix in $V$?

$$\|A\|^2 = \langle A, A\rangle = \text{trace}\left(A^T A\right) = \text{trace}\left(I_2\right) = 2$$

$$\therefore \boxed{\|A\| = \sqrt{2}}.$$

6. (4 points) Multiple choice. Choose the best answer to each of the following questions.

(a) If $T : \mathbb{R}^{3\times3} \to \mathbb{R}^{2\times2}$ is a linear transformation, then the kernel of $T$ must be at least

i. 1-dimensional.

ii. 2-dimensional.

iii. 3-dimensional.

iv. 4-dimensional.

$\boxed{\text{v. 5-dimensional.}}$

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim(\mathbb{R}^{3\times3}) = 9$$

$$\therefore \dim(\ker(T)) = 9 - \dim(\operatorname{im}(T))$$
$$\geq 9 - \dim(\mathbb{R}^{2\times2})$$
$$\geq 9 - 4 = \boxed{5}$$

(b) If $\vec{v}$ is a unit vector in $\mathbb{R}^n$, then the matrix $\vec{v}\vec{v}^T$ has rank

i. 0.

$\boxed{\text{ii. 1.}}$

iii. $n$.

iv. $n-1$.

v. cannot be determined from the given information.

The $n \times n$ matrix $\vec{v}\vec{v}^T$ represents projection onto the line spanned by $\vec{v}$. Therefore the image of $\vec{v}\vec{v}^T$ is one-dimensional.