MATH 2270

Exam #3 - Fall 2008

Name: Answer Key

1. (4 points) Find the determinant of the following matrix. Show your work.

$$\begin{pmatrix} 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Expand across the bottom row:

$$1 \cdot \det \begin{pmatrix} 0 & 3 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 4 \end{pmatrix} = \text{(expand down first column)}$$

$$= 1 \cdot (-2) \cdot \det \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 1 \cdot (-2) \cdot (12 - 1) = 1 \cdot (-2) \cdot 11 = \boxed{-22}$$

2. (6 points) Consider a 3×3 matrix A with rows $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$

$$A = \begin{pmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 & - \\ - & \vec{v}_3 & - \end{pmatrix}$$

and suppose $\det(A) = 8$.

(a) Find $\det \begin{pmatrix} 2\vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_2 \end{pmatrix}$.

- Multiplying the 1st row by 2 (Scales det(A) by 2)
- Swapping rows 2 and 3 (Scales det(A) by −1)

$\therefore$ we get $(-1) \cdot 2 \cdot \det(A) = \boxed{-16}$

(b) Find $\det(A^{-1}A^{T}A)$.

$$\begin{aligned} \det(A^{-1}A^{T}A) &= \det(A^{-1})\det(A^{T})\det(A) \\ &= \cancel{\det(A)^{-1}}\det(A^{T})\cdot\cancel{\det(A)} \\ &= \det(A^{T}) = \det(A) = \boxed{8} \end{aligned}$$

(c) Suppose the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are orthogonal and have norm x, i.e.

$$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = x.$$

Find x.

Since $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are orthogonal, they span a cube in $\mathbb{R}^3$.

We know this cube has volume $\det(A) = 8$, so each side has length $\boxed{2}$.

3. (8 points) For $\alpha \in \mathbb{R}$, let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & \alpha \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

and observe that the only eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

(a) What is the algebraic multiplicity of the eigenvalue $\lambda = 1$?

$\boxed{3}$

(b) What is the geometric multiplicity of the eigenvalue $\lambda = 1$?

geometric mult = $\dim \ker (A - I) = \dim\left(\ker \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & \alpha \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}\right)$

$= \boxed{2}$

(c) For which values of α will the geometric multiplicity of $\lambda = 2$ and the algebraic multiplicity of $\lambda = 2$ be equal?

Need $\dim\left(\ker \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right) = 2$. This is possible if and only if there are two rows of zeros, or $\alpha = 0$.

(d) If $\alpha = 0$, is A diagonalizable? Explain your answer.

No, since by (a) and (b) there is an eigenvalues whose geom. mult. is strictly less than its als. mult.

4. (8 points) Suppose

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}.$$

Find a diagonal matrix D and an orthogonal matrix S such that $D = S^T A S$. Show your work.

$$f_A(\lambda) = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)$$

$$E_5 = \ker \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$$E_{-5} = \ker \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

$\therefore$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \quad \text{and} \quad S = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

5. (10 points)

(a) Is the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ similar to the matrix $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$?

$\boxed{\text{No}}$ they don't have the same trace.

(b) Is the matrix $\begin{pmatrix} -1 & 6 \\ -2 & 6 \end{pmatrix}$ similar to the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$?

$f_A(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

$\boxed{\text{Yes}}$ because they have the same eigenvalues and they are all distinct.

(c) Suppose $\vec{v}$ is an eigenvector of an invertible $n \times n$ matrix A and let λ be the associated eigenvalue. Is $\vec{v}$ an eigenvector of $A^{-1} + 2I_n$? If so, what is the associated eigenvalue? If not, explain why.

$(A^{-1} + 2I_n)\vec{v} = A^{-1}\vec{v} + 2I_n\vec{v} = \frac{1}{\lambda}\vec{v} + 2\vec{v} = \boxed{\left(\frac{1}{\lambda}+2\right)\vec{v}}.$

Therefore, $\vec{v}$ is an eigenvector w/ eigenvalue $\boxed{\frac{1}{\lambda}+2}$.

(d) Let $q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 - 6x_1x_2 + 7x_2x_3$ be a quadratic form. Find a symmetric 3×3 matrix A such that $q(\vec{x}) = \vec{x}^T A\vec{x}$.

$$A = \begin{pmatrix} 3 & -3 & 0 \\ -3 & 4 & 7/2 \\ 0 & 7/2 & 5 \end{pmatrix}$$

(e) Suppose A is ~~an~~ symmetric orthogonal $n \times n$ matrix. Find trace(A^2).

The eigenvalues of A are ± 1, so the eigenvalues of A^2 are all $(\pm 1)^2 = 1$. Therefore, the trace of A^2 is the sum of the eigenvalues $= \boxed{n}$.

6. (4 points) Multiple choice. Choose the best answer to each of the following questions.

(a) If a 2×2 matrix A has eigenvalues $\lambda = 1$ and $\lambda = 2$, then trace($A^{\cancel{50}\,100}$) is

i. 2^{100}

ii. 2^{101}

$\boxed{\text{iii. } 2^{100} + 1}$

iv. 3^{100}

v. Cannot be determined from the given information.

$A = SDS^{-1}$

$A^{50} = SD^{50}S^{-1} = S \begin{pmatrix} 1^{50} & \\ & 2^{50} \end{pmatrix} S^{-1}$ ∴

$\text{trace}(A^{50}) = \text{trace} \begin{pmatrix} 1^{50} & \\ & 2^{50} \end{pmatrix} = 2^{50} + 1$

(b) If A is a symmetric positive definite 2×2 matrix, which of the following must be true?

i. The algebraic multiplicity of every eigenvalue is 1.

ii. A^{-1} is negative definite.

iii. $\det(-A) < 0$

$\boxed{\text{iv. The curve defined by the equation } \vec{x}^T A \vec{x} = 1 \text{ is an ellipse.}}$

v. none of the above.