MATH 2270

Quiz #5 - Fall 2008

DUE: In class Friday (10/31)

Name: Answer Key

NOTES:

- Work individually, but feel free to use books, notes, etc.
- In order to receive full credit on a problem, you must clearly show each step used to obtain the solution.
- The quiz is due at (or before) the *beginning* of class on Friday 10/31. If you cannot attend class on Friday (or choose not to), drop your quiz off at my office (JWB 213) or put it in my mailbox (JWB 228) sometime *before* the start of class.
- I will post the solutions on the course webpage

  http://www.math.utah.edu/~crofts

  immediately following class on Friday. Consequently, I cannot accept late quizzes.

1. (3 points) Let

$$V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

be a subspace of $\mathbb{R}^4$ and suppose $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$ with respect to the subspace V. Find $\vec{v}^{\parallel}$ and $\vec{v}^{\perp}$.

Let $\vec{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$, $\vec{u}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. Then

$$\vec{v}^{\parallel} = \operatorname{Proj}_V(\vec{v}) = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2)\vec{u}_2 + (\vec{v}\cdot\vec{u}_3)\vec{u}_3$$

$$= \frac{1}{4}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right)\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right)\cdot\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + \frac{1}{4}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{pmatrix}}$$

$$\vec{v}^{\perp} = \vec{v} - \vec{v}^{\parallel} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{pmatrix} = \boxed{\begin{pmatrix} 1/4 \\ -1/4 \\ 1/4 \\ -1/4 \end{pmatrix}}$$

<u>Note</u>: $\vec{v}^{\perp}$ is orthogonal to every element of V.

2. (3 points) Find the QR factorization of the matrix

$$M = \begin{pmatrix} 6 & 2 \\ 3 & -6 \\ 2 & 3 \end{pmatrix}.$$

Let $\vec{v}_1 = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$. Then $\|\vec{v}_1\| = 7$ and $\vec{u}_1 = \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix}$

Let $\vec{v}_2 = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$. Then

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - \overset{0}{\overset{\parallel}{(\vec{v}_2 \cdot \vec{u}_1)}} \cdot \vec{u}_1 = \vec{v}_2.$$

Let $\vec{u}_2 = \dfrac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \dfrac{\vec{v}_2}{7} = \begin{pmatrix} 2/7 \\ -6/7 \\ 3/7 \end{pmatrix}$.

Therefore:

$M = QR$, where $Q = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix}$ and $R = \begin{pmatrix} \|\vec{v}_1\| & \vec{v}_2 \cdot \vec{u}_1 \\ 0 & \|\vec{v}_2^{\perp}\| \end{pmatrix}$

So, $\boxed{Q = \begin{pmatrix} 6/7 & 2/7 \\ 3/7 & -6/7 \\ 2/7 & 3/7 \end{pmatrix} \text{ and } R = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}.}$

3. (2 points) Find the 3×3 matrix A of the orthogonal projection onto the line in $\mathbb{R}^3$ spanned by the vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Let Q be the 3×1 matrix $\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Then $A = QQ^T = \frac{1}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 4/9 & 4/9 \\ 2/9 & 4/9 & 4/9 \end{pmatrix}.$$

4. (3 points) True/False. Indicate whether the following statements are true or false.

(a) If the matrices A and B commute, then the matrices A^T and B^T also commute.

$$A^T B^T = (BA)^T = (AB)^T = B^T A^T \quad \checkmark$$

$\boxed{\text{True}}$

(b) If A is a square matrix, then $\frac{1}{2}(A - A^T)$ is a skew-symmetric matrix.

$$\tfrac{1}{2}(A - A^T)^T = \tfrac{1}{2}(A^T - A) = -\tfrac{1}{2}(A - A^T) \quad \checkmark$$

$\boxed{\text{True}}$

(c) There exists a subspace $V \subset \mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$.

We must have $\dim(V) + \dim(V^\perp) = 5$. Since $\dim(V) + \dim(V^\perp)$ must be integers, this is not possible.

$\boxed{\text{False}}$