LECTURE 11: PROBABILISTIC SOLUTION TO ELLIPTIC PDE'S

§1. ANOTHER ITÔ's FORMULA

We now explore some of the many connections between Brownian motion and second-order partial differential equations (PDE's). To start, we need a variant of Itô's formula. This one is an Itô-type development for a function $f(x,t)$ of space-time (x,t); the "space variable" is $x \in \mathbf{R}^d$, and the "time variable" is $t \geq 0$.

Throughout, W denotes d-dimensional Brownian motion.

(1.1) Another Itô's Formula. For any $T \geq t \geq 0$,

$$
\begin{aligned}
f(W(t), T-t) &= f(W(0), t) + \sum_{j=1}^{d} \int_0^T \frac{\partial}{\partial x_j} f(W(s), T-s)\, dW_j(s) \\
&\quad + \sum_{j=1}^{d} \int_0^T \frac{1}{2} \Delta f(W(s), T-s)\, ds + \int_0^T \frac{\partial}{\partial t} f(W(s), T-s)\, ds,
\end{aligned}
\tag{1.2}
$$

where $\Delta f(x,t) := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} f(x,t)$ is the *Laplacian* of f in the "space variable" $x \in \mathbf{R}^d$.

§2. THE HEAT EQUATION

The heat equation is the equation that governs the flow of heat in a nice medium. If $u(x,t)$ denotes the amount of heat at place $x \in \mathbf{R}^d$ at time t, then it states that u is "the continuous solution" to the following:

$$
\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= \frac{1}{2} \Delta u(x,t), \quad t \geq 0, x \in \mathbf{R}^d, \\
u(x,0) &= f(x), \qquad x \in \mathbf{R}^d,
\end{aligned}
\tag{2.1}
$$

where f is the function that tells us the initial amount of heat introduced at each point $x \in \mathbf{R}^d$ in space, and u tells us how this heat propagates (i.e., cooling). The number $\frac{1}{2}$ is chosen for the sake of convenience and can be replaced by any other number c; in general, this is the so-called *thermal conductivity* of the medium that is being heated, and can be obtained by a change of variables of type $v(x,t) := u(\sqrt{c}x, t)$. Indeed, note that $\frac{\partial}{\partial t} v(x,t) = \frac{\partial}{\partial t} u(ax,t)$ and $\frac{\partial^2}{\partial x_j^2} v(x,t) = c \frac{\partial^2}{\partial x_j^2} u(ax,t)$. So that v solves

$$
\begin{aligned}
\frac{\partial}{\partial t} v(x,t) &= c \Delta v(x,t), \quad t \geq 0, x \in \mathbf{R}^d, \\
v(x,0) &= f(x/\sqrt{c}), \qquad x \in \mathbf{R}^d.
\end{aligned}
\tag{2.2}
$$

So we might as well study (2.1) when the thermal conductivity is $\frac{1}{2}$.

(2.3) The Probabilistic Solution. *The solution to (2.2) can be written as follows, where W denotes d-dimensional Brownian motion: $u(x,T) = E_x\{f(W(T))\}$, where E_x denotes the expectation relative to Brownian motion started at $x \in \mathbf{R}^d$.*

(2.4) Itô's Formula Once More. We can deduce (2.3) from (1.2) with $T := t$ as follows:

$$\begin{aligned} u(W(T),0) = u(W(0),T) &+ \int_0^T -\frac{\partial}{\partial t} u(T-s, W(s))\, ds \\ &+ \int_0^T \frac{1}{2}\Delta u(T-s, W(s))\, ds + \text{stoch. integral.} \end{aligned} \tag{2.5}$$

All that we care about is that the expected value of the stochastic integral is zero; cf. the simulation approximation (2.9, Lecture 9) to convince yourselves of this. Moreover, the other two integrals are equal to $\int_0^T \left(\frac{1}{2}\Delta u - \frac{\partial}{\partial t}u\right) = 0$, since u solves the heat equation (2.1). So, we can take the expectation of (2.5) conditional on $W(0) = x$ (i.e., start your Brownian motion at $x \in \mathbf{R}^d$) to get $E_x\{u(W(T),0)\} = u(x,T)$. Since $u(y,0) = f(y)$ for all y, this proves (2.4). ♣

(2.6) Project. How would you simulate $E_x\{f(W(T))\}$? (Hint: Kolmogorov's strong law of large number (0.1, Lecture 2).)

(2.7) THE DIRICHLET PROBLEM. If you put a unit of charge in the middle of a sphere, it charges the outer shell of the sphere and the charge distribution is uniform. More generally, if D is a nice domain in $\mathbf{R}^d$ (the analogue of the sphere), and if f is the charge distribution on the boundary (or shell) ∂D of D, then we have a charge distribution $u(x)$ at x that is given by the Dirichlet problem:

$$\begin{aligned} \Delta u(x) = 0 \quad & x \in D \quad \text{i.e., no-flux inside} \\ u = f, \quad & \text{on } \partial D. \end{aligned} \tag{2.8}$$

The probabilistic solution, using Brownian motion, is $u(x) := E_x\{f(W(\tau_D))\}$, where W denotes Brownian motion started at x and in d dimensions, and τ_D is the first time W leaves D. How would you simulate this?