LECTURE 2: THE SIMPLE WALK IN DIMENSION ONE

Laws of large numbers are a class of results that state that, in one way or another, averaging many independent random quantities yields their expectation as long as you average enough things.

For example, suppose you wanted to know the average output α of a machine. If you could simulate the output of this machine on your computer, it would be natural to run several simulations, average the outputs, and declare that as an "estimate" for α. The following shows that this procedure actually works. You may need to refer to §2 of Lecture 1 for further motivation.

(0.1) Kolmogorov's Strong Law of Large Numbers. *Suppose $X_1, X_2, \ldots$ are independent and identically distributed. If $S_n := X_1 + \cdots + X_n$ denote the corresponding random walk, and if $\mu := E\{X_1\}$ exists, then*

$$P\left\{\lim_{n\to\infty} \frac{S_n}{n} = \mu\right\} = 1.$$

In the unbiased case where $\mu = 0$, this shows that the asymptotic value of the walk is much smaller than n. In fact, in most of these cases, the asymptotic value is of order $\sqrt{n}$.

(0.2) The Central Limit Theorem. *Suppose $X_1, X_2, \ldots$ are independent and identically distributed. If $S_n := X_1 + \cdots + X_n$ denote the corresponding random walk, and if $E\{X_1\} = 0$ and $0 < \sigma^2 := \operatorname{Var}(X_1) < +\infty$, then for any real number x,*

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\sqrt{n}} \le x\right\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-y^2/2\sigma^2}\, dy.$$

In the physics literature, this type of $\sqrt{n}$-growth is referred to as "diffusive."

§1. THE STRONG LAW FOR THE SIMPLE WALK

Once again, S_n is now the simple walk (on the integer lattice). While the general form of the Kolmogorov strong law is a rather difficult result, for the simple walk, things are not so bad as we shall see.

Here is a start: Let us apply Chebyshev's inequality from (4.4) of Lecture 1 to see that for any $\varepsilon > 0$,

$$P\{|S_n| \ge n\varepsilon\} \le \frac{\operatorname{Var}(S_n)}{n^2\varepsilon^2} = \frac{1}{n\varepsilon^2}. \tag{1.1}$$

We are using two more facts from Lecture 1. Namely, that the expectation of S_n is zero (3.1, Lecture 1) and its variance is n (3.2, Lecture 1). This shows that for *any* $\varepsilon > 0$ (however small),

$$\lim_{n\to\infty} P\left\{\left|\frac{S_n}{n}\right| \ge \varepsilon\right\} = 0.$$

This is not quite as strong as the strong law, but it has the right flavor. We will enhance this calculation to get the strong law.

Proof of The Strong Law For the Simple Walk: We can improve (1.1) by using higher moments than the second moment (i.e., the variance). Namely, let us use the Chebyshev inequality for fourth moments (4.5, Lecture 1) and the fact that $E\{S_n^4\} = 3n^2 - 2n \le 3n^2$ (3.3, Lecture 1) to obtain the following: For all $\varepsilon > 0$,

$$P\{|S_n| \ge n\varepsilon\} \le \frac{E\{S_n^4\}}{\varepsilon^4 n^4} \le \frac{3}{\varepsilon^2 n^2}. \tag{1.2}$$

So in fact the abov probability goes to zero faster than the rate of $(n\varepsilon^2)^{-1}$ stated in (1.1). Now let $\mathcal{N}$ denote the number of times the random walk is at least $n\varepsilon$ units away from the origin. That is,

$$\mathcal{N} := \sum_{n=1}^{\infty} \mathbf{1}_{\{|S_n| \ge n\varepsilon\}},$$

where $\mathbf{1}_A$ is the indicator of the event A; cf. (4.2, Lecture 1). Since $E\{\mathbf{1}_A\} = P\{A\}$, $E\{\mathcal{N}\} = \sum_{n=1}^{\infty} P\{|S_n| \ge n\varepsilon\}$. In particular, by (1.2) above, and using the fact that $1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \cdots$ is a summable sequence, we see that $E\{\mathcal{N}\} < +\infty$. This means that $\mathcal{N}$ is finite with probability one. In other words, we have shown that with probability one, for any $\varepsilon > 0$, there exists a random time $\mathcal{N}$ past which $|S_n| \le n\varepsilon$. This is the same as saying that with probability one, $S_n/n \to 0$. ♣

§2. RETURNS TO THE ORIGIN

What we have done is to show that S_n is much smaller than n as $n \to \infty$. One rough explanation for this is that S_n is fluctuating as $n \to \infty$; so much so that it has little time to go very far from the origin. This is one of the reasons that the movement of the simple walk has proven to be an important model for "one-dimensional molecular motion." (The more realistic three-dimensional setting will be covered soon.)

One way in which we can study the said fluctuation phenomenon more precisely, is by considering the notion of *recurrence*. In the context of nonrandom dynamical systems, this notion is due to the work of H. Poincaré.

Remember that S_0 is zero. That means that the random walk always starts at the origin. So it makes sense to consider N_n which is the number of returns to the origin by time n; i.e.,

$$N_n := \sum_{j=1}^{n} \mathbf{1}_{\{S_j = 0\}}, \qquad n = 1, 2, 3, \ldots.$$

(2.1) The Expected Number of Returns. *As $n \to \infty$, $E\{N_n\} \sim \sqrt{2n/\pi}$, where $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$.*

Proof: Note that

$$E\{N_n\} = E\left[\sum_{j=1}^{n} \mathbf{1}_{\{S_j=0\}}\right] = \sum_{j=1}^{n} P\{S_j = 0\}.$$

So it suffices to estimate $P\{S_j = 0\}$ for $j \to \infty$. First, we note that if j is an odd number $S_j \neq 0$. So it suffices to estimate $E\{N_n\}$ for n even. Moreover, if n is even,

$$E\{N_n\} = \sum_{j=1}^{n/2} P\{S_{2j} = 0\}.$$

Here is where combinatorics come in: Thanks to (1.1, Lecture 1), the probability that $S_{2j} = 0$ is equal to 2^{-2j} times the number of paths of length $2j$ such that at time j the path is at 0. Any such path $\pi_0, \ldots, \pi_{2j}$ hits 0 at time j if and only if it has gone to the right exactly j times, and gone to the left exactly j times. There are $\binom{2j}{j}$-many ways for choosing where these rights and lefts are, so

$$P\{S_{2j} = 0\} = 2^{-2j}\binom{2j}{j}.$$

This and the preceding display, together show

$$E\{N_n\} = \sum_{j=1}^{n/2} 2^{-2j}\binom{2j}{j}. \tag{2.2}$$

But $\binom{2j}{j} = (2j)!/(j!)^2$, and this can be estimated by

(2.3) Stirling's Formula. *As* $k \to \infty$, $k! \sim \sqrt{2\pi} k^{k+\frac{1}{2}} e^k$.

We use this to see that

$$\begin{aligned} E\{N_n\} &\sim \sum_{j=1}^{n/2} 2^{-2j} \frac{\sqrt{2\pi}(2j)^{2j+\frac{1}{2}} e^{-2j}}{\left(\sqrt{2\pi} j^{j+\frac{1}{2}} e^{-j}\right)^2} = \sum_{j=1}^{n/2} \frac{1}{\sqrt{2\pi}} \frac{2^{\frac{1}{2}}}{j^{\frac{1}{2}}} = \sqrt{\frac{1}{\pi}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} \\ &= \sqrt{n} \cdot \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\sqrt{j/n}}. \end{aligned} \tag{2.4}$$

But $\frac{1}{n}\sum_{j=1}^{nT} f(j/n) \to \int_0^T f(x)\,dx$ if f is continuous; in fact this is the Riemann-sum approximation of the calculus of real functions. Apply this with $f(x) := 1/\sqrt{x}$ to see that $(1/n)\sum_{j=1}^{n} 1/\sqrt{j/n} \sim \sqrt{n} \cdot \int_0^{1/2} 1/\sqrt{x}\,dx = \sqrt{2n}$. Together with (2.4), this completes our asymptotic evaluation of $E\{N_n\}$. ♣

§3. THE REFLECTION PRINCIPLE

Here is another application of the combinatorial way of thinking. This is a deep result from the 1887 work of D. André:

(3.1) The Reflection Principle. *For any $\lambda, n = 1, 2, \ldots$,*

$$P\left\{\max_{1\leq j\leq n} S_j \geq \lambda\right\} = 2P\{S_n \geq \lambda\}.$$

Proof: The combinatorial representation of the simple walk (1.1, Lecture 1) tells us that the above is *equivalent* to showing that

$$\begin{aligned}(\mathbf{3.2})\qquad & \#\,\{\text{paths that go over } \lambda \text{ before time } n\} \\ & \qquad = 2 \times \#\,\{\text{paths that are go over } \lambda \text{ at time } n\}\,.\end{aligned}$$

There are two types of paths that go over λ before time n: The first are those that are over λ at time n, i.e., those paths for which $\pi_n \geq \lambda$ (*Type 1*). The second (*Type 2*) are those that go over λ some time before time n and then go below it so that at time n, $\pi_n < \lambda$. If you think about it for a moment, you will see that (3.2) is really stating that the number of paths of *Type 2* is equal to the number of paths of *Type 1*. But this is clear from a picture; for example, see the picture at

`http://www.math.utah.edu/~davar/REU-2002/notes/lec2.html`.

Namely, any path of *Type 2*, can be reflected about the line $y = \lambda$ at the first time it hits λ. This gives a paths of *Type 1*. Conversely, any paths of *Type 1* can be reflected to give a path of *Type 2*. This shows that there are as many paths of each type, and we are done. ♣

§4. APPENDIX: STIRLING'S FORMULA

It would be a shame for you not to see why Stirling's formula (2.3 above) is true; so I have added this section to explain it, although we did not discuss this section's material in our meeting.

Consider $\ln(k!) = \sum_{i=2}^{k} \ln(i)$. By the integral test of calculus,

$$\int_1^k \ln(x)\,dx \leq \ln(k!) \leq \int_1^{k+1} \ln(x)\,dx.$$

But $\int_1^T \ln(x)\,dx = T\ln(T) - 1$. Therefore,

$$k\ln(k) - 1 \leq \ln(k!) \leq (k+1)\ln(k+1) - 1. \tag{4.1}$$

Now, recall Taylor's expansions for $\ln(1+y)$:

$$\ln(1+y) = 1 + y - \frac{y^2}{2} + \cdots. \tag{4.2}$$

We don't apply this to $\ln(k+1)$ but rather note that $\ln(k+1) = \ln(k) + \ln((k+1)/k)) = \ln(k) + \ln(1+\frac{1}{k})$. Apply (4.2) with $y = \frac{1}{k}$ to deduce that

$$\ln(k+1) = \ln(k) + \frac{1}{k} - \frac{1}{2k^2} + \cdots.$$

Put this back in to (4.1) to get

$$\begin{aligned} k\ln(k) \leq \ln(k!) &\leq (k+1)\left[\ln(k) + \frac{1}{k} - \frac{1}{2k^2} + \cdots\right] \\ &\leq (k+1)\left[\ln(k) + \frac{1}{k}\right] \\ &= k\ln(k) + \ln(k) + 1 + \frac{1}{k}. \end{aligned}$$

Since the exponential of $k\ln k$ is k^k, we can exponentiate the above inequalities to obtain

$$k^k \leq k! \leq k^{k+1} \times e^{1+\frac{1}{k}} \sim ek^{k+1}.$$

Stirling's formula is a much sharper version of these bounds. (For instance note that both sides are off by $k^{\frac{1}{2}}$ to the leading order.)